A Correct Set of Equations for the Real-time Ellipse Hough Transform Algorithm

Armando M. Fernandes

CENTRIA – Centre for Artificial Intelligence,
Universidade Nova Lisboa, Quinta da Torre, 2829-516 Caparica, Portugal
Email: arm.fernandes@gmail.com

Abstract— The Real-time Ellipse Hough Transform (RTEHT), by Zhang and Liu, which uses two bi-dimensional accumulators and one uni-dimensional accumulator to detect ellipses, was published in 2005. However, the published equations contain several errors that prevent users of RTEHT from obtaining correct ellipse parameters. This paper presents a corrected set of equations.

Keywords— Artificial intelligence, Pattern recognition, Hough Transform, Ellipse.

1 INTRODUCTION

The Hough transform allows for the identification of ellipses in images using accumulators that collect information on ellipse parameters such as position, size and rotation. This information is calculated using the positions of image points and the slopes of the curves to which the points belong. Aguado et al. [1] introduced the *K*-*N* accumulator in the Hough transform. Parameter *N* specifies the ratio of the sizes of the ellipse axes, and *K* specifies the tangent of the rotation angle of the main ellipse axis. Recently, Zhang and Liu [2, 3] presented a new algorithm called the Real-time Ellipse Hough Transform (RTEHT) that obtains information for the *K*-*N* accumulator using an innovative method. RTEHT selects four points in an image and uses them to calculate two possible values for parameter *K* and the associated two values for *N*. The two sets of parameters *K* and *N* correspond to a unique ellipse, and to two points in the *K*-*N* accumulator. This is a major improvement to the Hough transform because previously, the *K* values were given by a curve that was a function of *N* [1]. Consequently, a large number of *K*-*N* combinations had to be added to the accumulator.

Our attempt to reproduce RTEHT revealed that the set of equations provided by Zhang and Liu [2] does not yield correct results. Other authors, namely Mai et al. [4], have also reported problems when attempting to reproduce the work of Zhang and Liu [2]. Consequently, we derived a new and correct set of equations that we present here. The derivation procedure is similar to that of Aguado et al. [1] and Zhang and Liu [2]. We found errors in equations used to determine the ellipse centre *yy* coordinate, to derive parameter K, and in parameters used to determine the length of the *xx* component of the main ellipse axis.

The second section of this paper is dedicated to the derivation of the equations. A third section contains the description of the steps required to implement RTEHT with the correct equations, while a fourth section contains a numerical example of the implementation of the RTEHT.

2 NEW EQUATIONS

2.1 Geometry and auxiliary equations

The geometry under analysis is the one shown in Figure 1. The figure shows how to determine an ellipse center using four points. This result was presented by Yuen et al. [5] and used in Aguado et al. [6]. The relative positions of points P_1, P_2, P_3, and P_4 are irrelevant; however, the line segment joining points P_1 and P_2 cannot be parallel to that joining P_3 and P_4. The midpoint M_{12} between points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, has *xx* coordinate M_{12x} and *yy* coordinate M_{12y}. These coordinates are calculated using the equations:

$$M_{12x} = \frac{x_1 + x_2}{2}, \qquad (1)$$

$$M_{12y} = \frac{y_1 + y_2}{2}. \qquad (2)$$

Point T_{12} is the intersection of the lines tangent to the ellipse in points P_1 and P_2. Since T_{12} belongs to the lines passing over points P_1 and P_2, we can write, using the parametric equation of a line, that:

$$\phi_1 T_{12x} + b_1 = \phi_2 T_{12x} + b_2 \Leftrightarrow$$

$$T_{12x} = \frac{b_2 - b_1}{\phi_1 - \phi_2}, \qquad (3)$$

where ϕ_i is the slope, and b_i the y-intercept of the lines tangent to the ellipse in point P_i. From the parametric equation of a line:

$$b_1 = -\phi_1 x_1 + y_1. \qquad (4)$$

Consequently:

$$T_{12x} = \frac{-\phi_2 x_2 + y_2 + \phi_1 x_1 - y_1}{\phi_1 - \phi_2}. \qquad (5)$$

Using the equation for a line passing through points T_{12} and P_1, we get:

$$T_{12y} - y_1 = \phi_1 (T_{12x} - x_1) \Leftrightarrow$$
$$T_{12y} = -\frac{\phi_1\phi_2 (x_2 - x_1) + \phi_2 y_1 - \phi_1 y_2}{\phi_1 - \phi_2}. \qquad (6)$$

In Zhang and Liu [2], the equation for T_{12y} does not have the minus sign before the ratio.

The slope of the line joining points P_1 and P_2 is :

$$q_1 = \frac{y_2 - y_1}{x_2 - x_1}. \qquad (7)$$

The slope of the line joining T_{12} and M_{12} is:

$$q_2 = \frac{T_{12y} - M_{12y}}{T_{12x} - M_{12x}} =$$
$$= \frac{2\phi_1\phi_2 (x_2 - x_1) + (\phi_1 + \phi_2)(y_1 - y_2)}{2(y_1 - y_2) + (\phi_1 + \phi_2)(x_2 - x_1)}. \qquad (8)$$

In Zhang and Liu [2], the plus sign after the first term in parentheses in the denominator of q_2, is presented as a minus sign. In the following sections, the equations contain variables M_{34x}, M_{34y}, T_{34x}, T_{34y}, q_3, and q_4, that are calculated from points P_3 and P_4 in the same manner in which M_{12x}, M_{12y}, T_{12x}, T_{12y}, q_1, and q_2 are calculated from points P_1 and P_2.

2.2 Equations for ellipse centre determination

To determine the ellipse centre (X_c,Y_c), we must observe that line L_{12} passing through M_{12} and line L_{34} passing through M_{34} have the ellipse centre point in common. Therefore:

$$\begin{aligned} Y_c &= q_2\left(X_c - M_{12x}\right) + M_{12y} = \\ &= q_4\left(X_c - M_{34x}\right) + M_{34y}. \end{aligned} \qquad (9)$$

By solving, in order to X_c, the equation composed of the two terms to the left and to the right of the second equal sign, we get the following expression:

$$X_c = \frac{q_2 M_{12x} - q_4 M_{34x} - M_{12y} + M_{34y}}{q_2 - q_4}. \qquad (10)$$

Then:

$$\begin{aligned} Y_c &= q_2\left(X_c - M_{12x}\right) + M_{12y} = \\ &= \frac{q_2 M_{34y} - q_4 M_{12y} + q_2 q_4\left(M_{12x} - M_{34x}\right)}{q_2 - q_4}. \end{aligned} \qquad (11)$$

In Zhang and Liu [2], the variables in parentheses have opposite signs to the ones shown here.

2.3 Equations for the *K*-*N* accumulator

In the Hough transform, it is necessary to define two parameters, K and N, which are given by:

$$K = \tan \rho, \qquad (12)$$

$$N = \frac{B}{A}, \qquad (13)$$

where ρ is the angle of counter-clockwise rotation of the main ellipse axis of size $2A$, and N is the ratio between the two ellipse semi-axes sizes A and B. In our paper, the main ellipse axis may be smaller than the other axis. Zhang and Liu [2] present an equation that specifies N in terms of K. This equation was introduced by Aguado et al. [1]:

$$N^2 = -\frac{(q_1 - K)(q_2 - K)}{(1 + q_1 K)(1 + q_2 K)}. \qquad (14)$$

Zhang and Liu [2] used equation (14) to derive an equation for K that is independent of N. The main idea is that equation (14) is valid for any two points. Therefore, we can write:

$$-\frac{(q_1 - K)(q_2 - K)}{(1 + q_1 K)(1 + q_2 K)} = -\frac{(q_3 - K)(q_4 - K)}{(1 + q_3 K)(1 + q_4 K)}. \qquad (15)$$

After some algebra, we get the following equation:

$$\begin{aligned} &\alpha\left(K^4 - 1\right) + \beta\left(K + K^3\right) = 0 \Leftrightarrow \\ &\alpha\left(K^2 - 1\right)\left(K^2 + 1\right) + \beta K\left(1 + K^2\right) = 0 \Leftrightarrow \\ &K^2 + 1 = 0 \vee \alpha K^2 + \beta K - \alpha = 0 \end{aligned} \qquad (16)$$

where:

$$\alpha = (q_1 q_2 - q_3 q_4), \qquad (17)$$

$$\begin{aligned} \beta = q_2 q_4 (q_3 - q_1) + q_1 q_3 (q_4 - q_2) + \\ + (q_1 + q_2 - q_3 - q_4). \end{aligned} \qquad (18)$$

The solution with K equal to $\pm\sqrt{-1}$ is impossible, but the solution of the second equation in (16) is:

$$K = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha^2}}{2\alpha}. \qquad (19)$$

Equation (19) yields two solutions for K that correspond to the tangent of angle ρ and the tangent of the angle $\rho+\pi/2$. The two solutions correspond to the same ellipse, and each one may be determined with the '+' or the '–' sign in equation (19), depending on the values of α and β. The equation for K provided by Zhang and Liu [2] is:

$$K_{Z\&L} = \pm\sqrt{1-\frac{\beta}{\alpha}} \Leftrightarrow$$
$$\alpha K_{Z\&L}^2 + \beta - \alpha = 0. \qquad (20)$$

The solutions of equations (20) and (16) are equal only when K is one. Therefore, Zhang and Liu's [2] equation for K is valid only for an ellipse rotated counter-clockwise by $\pi/4$.

2.4 Equations for determination of the main axis size

To determine the xx component of the main ellipse axis (A_x) we must use the equation provided by Aguado et al. [1] and Zhang and Liu [2] that is valid for unrotated ellipses centered on the origin:

$$A_x = \sqrt{\frac{x_0^2 N^2 + y_0^2}{N^2\left(1+K^2\right)}}, \qquad (21)$$

with x_0 and y_0 being the coordinates of an ellipse point. To apply equation (21) with any ellipse rotated counter-clockwise by an angle ρ and centered at (X_c,Y_c), we must translate points (x_i,y_i) to the origin and rotate them clockwise by an angle

ρ. In two dimensions, the rotation and translation is done in the following way:

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos\rho & \sin\rho \\ -\sin\rho & \cos\rho \end{bmatrix} \begin{bmatrix} x_i - X_c \\ y_i - Y_c \end{bmatrix}. \qquad (22)$$

Using the derivation of Aguado et al. [1], we know that:

$$K = \tan\rho = \frac{A_y}{A_x} = \frac{\sin\rho}{\cos\rho}, \qquad (23)$$

which implies that:

$$\sin^2\rho = K^2 \cos^2\rho. \qquad (24)$$

We can now derive the equation for cosine of ρ:

$$\begin{aligned} &\sin^2\rho + \cos^2\rho = 1 \Leftrightarrow \\ &K^2\cos^2\rho + \cos^2\rho = 1 \Leftrightarrow \\ &\cos\rho = \frac{1}{\sqrt{1+K^2}}. \qquad (25) \end{aligned}$$

Consequently:

$$\sin\rho = \frac{K}{\sqrt{1+K^2}}. \qquad (26)$$

We now have everything that is needed to calculate, with the linear system of equations (22), the parameters x_0 and y_0 that are required to determine A_x:

$$x_0 = \frac{x_i - X_c}{\sqrt{K^2+1}} + \frac{(y_i - Y_c)K}{\sqrt{K^2+1}}, \qquad (27)$$

$$y_0 = -\frac{(x_i - X_c)K}{\sqrt{K^2+1}} + \frac{y_i - Y_c}{\sqrt{K^2+1}}. \qquad (28)$$

In Aguado et al. [1] and in Zhang and Liu [2], the first ratio of the sum in equation (28) has the opposite sign.

3 PROCEDURE TO DETERMINE ELLIPSE PARAMETERS

The equations presented in this work correspond to a Hough transform and allow the determination of ellipse parameters by using accumulators. To determine ellipse parameters, we have to go through the following steps:

- Choose four points.
- Calculate q_1, q_2, q_3, q_4 using equations (7) and (8). This requires parameter ϕ_i of the four chosen points.
- Calculate X_c using equation (10) and Y_c using equation (11). This requires the values of q_2 and q_4 and the coordinates of points M_{12} and M_{34}.
- Calculate K using equation (19). Parameters q_1, q_2, q_3, and q_4 are required.
- Calculate N using equation (14). Parameters q_1, q_2, and K are required.
- Determine x_0 and y_0 using equations (27) and (28). This can be done using any of the four points selected initially. We must know X_c, Y_c and K.
- Determine A_x using equation (21). This uses K, N, x_0 and y_0.
- With K and equation (12), calculate the ellipse rotation angle ρ.
- Use the angle ρ and A_x to get A.
- Insert the values of N and A into equation (13) to find B.

To find ellipses in a binary image, we must repeat the above steps several times. For each group of four points selected, we obtain point (X_c,Y_c), point (K,N) and point A_x. Each of these three points is to be added to an accumulator: in the first two cases, a bi-dimensional one, and in the last case a uni-dimensional one. The parameters of the ellipses in the analyzed image correspond to the values of the accumulator bins with the largest frequencies.

4 NUMERICAL TESTS

4.1 Generating ellipse points and their slopes for exemplification

To demonstrate that the equations presented in the previous sections give the correct results, we will present, as an example, the calculations for an ellipse. We must determine the coordinates of the ellipse points (x_i,y_i) as well as the slopes ϕ_i of the lines tangent to the ellipse at those points, because these are the inputs of RTEHT. Equations (31) and (32) for y_i and ϕ_i, respectively, shown below, are for exemplification purposes. In a normal case of ellipse detection the point coordinates are determined directly from the image in which the ellipses are being detected, and the slope is that of the line tangent to the curve to which the points belong.

We start with the equation:

$$\frac{\left[(x-X_c)\cos\rho+(y-Y_c)\sin\rho\right]^2}{A^2}+ \\ +\frac{\left[-(x-X_c)\sin\rho+(y-Y_c)\cos\rho\right]^2}{B^2}=1, \quad (29)$$

which corresponds to an ellipse with semi-axes of sizes A and B, centered on position (X_c,Y_c) and whose axes are rotated counter-clockwise by an angle ρ. This angle corresponds to the rotation of the main ellipse axis, whose size is $2A$, relative to a horizontal axis. In equation (29), the rotation is clockwise (as presented in the linear system of equations (22)), to make the main ellipse axis "return" to the horizontal. To determine the point coordinates, we transform equation (29) into:

$$\mu(y - Y_c)^2 + \varepsilon(y - Y_c) + \sigma = 0, \qquad (30)$$

whose solution is:

$$y - Y_c = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4\mu\sigma}}{2\mu} \wedge \varepsilon^2 - 4\mu\sigma > 0, \qquad (31)$$

where:

$$\mu = \frac{\sin^2 \rho}{A^2} + \frac{\cos^2 \rho}{B^2},$$
$$\varepsilon = 2\left(\frac{1}{A^2} - \frac{1}{B^2}\right)(x - X_c)\cos\rho \sin\rho,$$
$$\sigma = \left(\frac{\cos^2 \rho}{A^2} + \frac{\sin^2 \rho}{B^2}\right)(x - X_c)^2 - 1.$$

To calculate the slope of the lines tangent to the ellipse, ϕ_i, we must calculate the derivative of equation (31):

$$\phi = \frac{dy}{dx} = \frac{1}{2\mu}\left[-\frac{d\varepsilon}{dx} \pm \frac{1}{\sqrt{\varepsilon^2 - 4\mu\sigma}}\left(\varepsilon\frac{d\varepsilon}{dx} - 2\mu\frac{d\sigma}{dx}\right)\right], \qquad (32)$$

where:

$$\frac{d\varepsilon}{dx} = 2\left(\frac{1}{A^2} - \frac{1}{B^2}\right)\cos\rho \sin\rho,$$
$$\frac{d\sigma}{dx} = 2\left(\frac{\cos^2 \rho}{A^2} + \frac{\sin^2 \rho}{B^2}\right)(x - X_c).$$

4.2 Data used in the exemplification

For our example, we defined two ellipses whose parameters are shown in Table 1. In Table 2 and Table 3, find the coordinates, (x_i,y_i), of five groups of four points from the two ellipses specified in Table 1. The slopes, ϕ_i, of the lines tangent to the ellipse in those points are also shown. In the calculations, the values of x_i, that were needed to generate y_i and ϕ_i using equations (31) and (32), respectively, had five significant digits.

The values of the parameters in Table 1 were introduced into the calculations with all significant digits that the software could use, even though the values shown in the table present a maximum of six significant digits.The values for the various parameters resulting from the calculations are presented with five significant digits. The rounding off was done only at the end of all calculations.

4.3 Results with previously published RTEHT equations

In this section, we present calculations with the equations from Zhang and Liu [2]. The parameters shown in Table 4 and Table 5 are the following:

- *NZhang* – Parameter *N*, calculated with equation (14), which is equal here and in the work of Zhang and Liu. The *K* value used is arctan(-π/6) for ellipse number 1 and arctan(π/5) for ellipse number 2, from Table 1, and q_2 is calculated from Zhang and Liu's equations.
- X_c*Zhang* e Y_c*Zhang* – Coordinates of the ellipse centre determined using both equations and values for q_2 and q_4 from Zhang and Liu.
- Y_c*Zhang2*- The *yy* coordinate of the ellipse centre determined using the

equation of Zhang and Liu and values for q_2 and q_4 determined with equation (8) from our work.

- *KZhang* – Parameter K determined with equations and values for q_2 and q_4 from Zhang and Liu.
- *KZhang2* - Parameter K calculated using the equation of Zhang and Liu for K and values for q_2 and q_4 determined with equation (8).
- *A_xZhang* – Parameter A_x determined using y_0 from Zhang and Liu. K and N values used are those from Table 1, namely arctan(-π/6) and 68 divided by 101.6 for ellipse number 1 and arctan(π/5) and 0.5 for ellipse number 2.
- For X_c and Y_c, the values we used were also from Table 1. The equation for A_x in Zhang and Liu and in the present paper is equal.

The results in Table 4 and Table 5 were obtained in two situations: when the values used for the input parameters of Zhang and Liu's equations were the correct ones, and when those values were calculated with Zhang and Liu's equations. In both tables the values shown for the ellipse parameters do not correspond to the expected values shown in Table 1, and in some cases those values are even complex numbers. Consequently, these results allow us to conclude that the equations for q_2, Y_c, K and y_0 provided by Zhang and Liu do not allow calculation of the correct ellipse parameters for an arbitrary ellipse.

4.4 Results with our equations

Finally, in Table 6 and Table 7, we show that for the points from Table 2 and Table 3, respectively, we can get the correct ellipse parameters shown in Table 1 by using our equations. In the table, K_{plus} and K_{minus} were calculated from equation (19) using the '+' and '–' signs, respectively. Parameter N_{plus} was calculated with K_{plus} and N_{minus} was calculated with K_{minus}. A_{xplus} was calculated with K_{plus} and N_{plus}, whereas A_{xminus} was calculated with N_{minus} and K_{minus}. Therefore, there are two possible sets of ellipse parameters: the first one is N_{plus}, K_{plus}, and A_{xplus} and the second one is N_{minus}, K_{minus}, and A_{xminus}.

5 Conclusion

We have presented a set of equations that can determine the correct ellipse parameters when four points of that ellipse are used. These equations define the algorithm RTEHT, which is a Hough transform with three accumulators. We provided a numerical example that shows the correctness of our set of equations, and demonstrates that the equations presented in the original work that describes RTEHT do not yield the correct ellipse parameters. The numerical examples and the detailed description of the steps required to implement RTEHT provided in this paper are helpful for someone trying to implement a Hough transform for the first time. The proposed changes to the equations do not affect RTEHT capability to process images in real time. The reason is that the changes do not concern the method that allows this capability, i.e., the method for selecting groups of four points for determination of ellipse parameters.

We cannot determine quantitatively the differences in detection efficiency between the original and the corrected set of equations because we are unable to reproduce the results reported in the paper of Zhang and Liu that presents RTEHT.

In fact, we cannot detect ellipses when RTEHT's original set of equations is used. An evaluation of RTEHT's detection efficiency when the new set of equations is employed is beyond the scope of the present paper, but it is an interesting subject for future work.

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Figure 1 – Geometry for the determination of an ellipse centre using four points.

	Ellipse 1		Ellipse 2	
Xc	-10.5		-100	
Yc	150.3		-50	
A	101.6	68	20	10
B	68	101.6	10	20
ρ	$-\pi/6$	$-\pi/6+\pi/2$	$\pi/5$	$\pi/5+\pi/2$
K=arctan (ρ)	-0.577352	1.73205	0.726543	-1.37638
N= (B/A)	0.669291	1.49412	0.5	2
A_x	87.9882	34	16.1803	5.87785

Table 1 - Parameters of an ellipse used to exemplify the use of our equations. The two sets of parameters, in different columns, represent the same ellipse.

Example	1	2	3	4	5
x_1	-65.364	-4.4040	-4.4040	-51.140	-95.844
y_1	225.09	221.70	221.70	95.475	205.16
ϕ_1	0.27785	-0.32760	-0.32760	-0.64801	1.3720
x_2	56.556	9.8200	-65.364	50.460	-55.204
y_2	183.22	73.143	225.09	77.502	227.19
ϕ_2	-1.0621	-0.10604	0.27785	0.38022	0.14056
x_3	-51.140	-103.97	56.556	78.908	-51.140
y_3	95.475	186.08	183.22	102.16	95.475
ϕ_3	-0.64801	5.4375	-1.0621	2.0314	-0.64801
x_4	70.780	36.236	-51.140	56.556	-10.500
y_4	90.591	200.96	95.475	183.22	77.058
ϕ_4	1.0411	-0.72020	-0.64801	-1.0621	-0.27732

Table 2 - Ellipse point coordinates, (x_i,y_i), and ellipse slopes, ϕ_i , at those points. The data is for five sets of four points from ellipse number1.

Example	1	2	3	4	5
x_1	-113.40	-97.400	-94.600	-92.400	-115.40
y_1	-49.157	-37.264	-36.369	-35.917	-52.221
ϕ_1	1.3182	0.37827	0.25843	0.14933	1.8322
x_2	-101.40	-92.000	-108.20	-83.400	-108.00
y_2	-39.095	-56.436	-43.732	-38.932	-64.138
ϕ_2	0.53645	0.83557	0.84698	-1.9755	0.12719
x_3	-93.400	-117.00	-115.20	-115.20	-107.40
y_3	-36.093	-56.354	-62.771	-62.771	-43.073
ϕ_3	0.20124	4.7125	-0.78793	-0.78793	0.80268
x_4	-116.00	-89.400	-100.00	-105.40	-96.000
y_4	-61.989	-35.743	-61.618	-41.568	-59.374
ϕ_4	-1.2185	-0.046003	0.48138	0.70433	0.64260

Table 3 – Ellipse point coordinates, (x_i,y_i), and ellipse slopes, ϕ_i , at those points.

The data is for five sets of four points from ellipse number 2.

Example	1	2	3	4	5
NZhang	0.031772	0.67905	0.41612	0.26563	1.1587 j
X_c*Zhang*	31.039	-54.109	3.862	-87.803	-718.1
Y_c*Zhang*	198.06	145.69	262.4	142.18	106.3
Y_c*Zhang2*	89.516	132.22	64.092	136.84	218.84
KZhang	4.089	3.1811	1.2603 j	3.1775	3.9353 j
KZhang2	1.4679	1.4679	1.4679	1.4679	1.4679
A_x*Zhang*	140.15	80.497	80.497	35.784	146.02

Table 4 - Results using the equations of Zhang and Liu [2] for ellipse number 1.

"j" represents $\sqrt{-1}$.

Example	1	2	3	4	5
NZhang	0.21551	0.43577 j	0.28703 j	1.6144 j	3.0904 j
X_c*Zhang*	-113.52	-64.171	-112.10	-73.849	-133.65
Y_c*Zhang*	-44.570	-65.860	-39.939	-50.931	63.501
Y_c*Zhang2*	-51.483	-43.179	-66.232	-61.835	-539.54
KZhang	1.2063	0.79706	3.4826 j	3.4215 j	1.0222
KZhang2	0.59174	0.59174	0.59174	0.59174	0.59174
A_x*Zhang*	14.337	20.657	25.067	28.192	20.788

Table 5 – Results using the equations of Zhang and Liu [2] for ellipse number 2.

"j" represents $\sqrt{-1}$.

Example	1	2	3	4	5
X_c	-10.5	-10.5	-10.5	-10.5	-10.5
Y_c	150.3	150.3	150.3	150.3	150.3
K_{plus}	-0.57735	1.732	1.732	1.732	1.732
K_{minus}	1.732	-0.57735	-0.57735	-0.57735	-0.57735
N_{plus}	0.66929	1.4941	1.4941	1.4941	1.4941
N_{minus}	1.4941	0.66929	0.66929	0.66929	0.66929
A_{xplus}	87.988	34	34	34	34
A_{xminus}	34	87.988	87.988	87.988	87.988

Table 6 - Results for ellipse number 1 determined with our equations. K_{plus} and K_{minus} correspond to equation (19) with the '+' and '–' signs, respectively.

Example	1	2	3	4	5
X_c	-100	-100	-100	-100	-100
Y_c	-50	-50	-50	-50	-50
K_{plus}	-1.3764	-1.3764	-1.3764	-1.3764	-1.3764
K_{minus}	0.72654	0.72654	0.72654	0.72654	0.72654
N_{plus}	2	2	2	2	2
N_{minus}	0.5	0.5	0.5	0.5	0.5
A_{xplus}	5.8779	5.8779	5.8779	5.8779	5.8779
A_{xminus}	16.180	16.180	16.180	16.180	16.180

Table 7 – Results for ellipse number 2 determined with our equations. K_{plus} and K_{minus} correspond to equation (19) with the '+' and '–' signs, respectively.

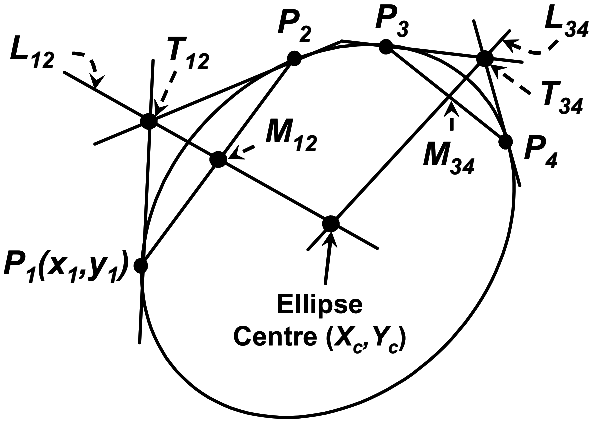

Figure 1