Behavioral algebraization of da Costa's $\mathcal{C}$-systems

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ABSTRACT. *It is well-known that da Costa's $\mathcal{C}$-systems of paraconsistent logic do not admit a Blok-Pigozzi algebraization. Still, an algebraic flavored semantics for them has been proposed in the literature, namely using the class of so-called da Costa algebras. However, the precise connection between these semantic structures and the $\mathcal{C}$-systems was never established at the light of the theory of algebraizable logics. In this paper we propose to study the $\mathcal{C}$-systems from an algebraic point of view, and to fill in this gap by using the tools and techniques of the newly developed behavioral approach to abstract algebraic logic. As a by-product of the approach, we also rediscover the bivaluation semantics of the logics.*

KEYWORDS: *$\mathcal{C}$-systems, algebraization of logics, behavioral reasoning, da Costa algebra.*

1. Introduction

The roots of the theory of algebraization of logics can be traced back to the work of Tarski. He was the first to establish precisely the connection between Boolean algebras and classical propositional logic. His approach built on Lindenbaum's idea of viewing the set of formulas of the logic as a free algebra with operations induced by the connectives. Logical equivalence then becomes a congruence relation on this algebra, that is, an equivalence relation compatible with every operation of the algebra. Moreover, the quotient algebra obtained by factoring the original algebra of formulas by logical equivalence is precisely a Boolean algebra. This construction, which became known as the Lindenbaum-Tarski algebraization method, was then successfully applied to obtain algebraic counterparts for a number of different logics. Nevertheless,

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it was soon realized that this technique is not very general, since logical equivalence is not always a congruence. The theory of algebraization of logics then evolved in the direction of finding a general congruence on a logic in such a way that a similar quotient construction could be obtained. Blok and Pigozzi gave, in (Blok *et al.*, 1989), the first precise abstract definition of the notion of algebraizable logic. The general theory of algebraization of logics, abstract algebraic logic, or simply AAL, from now on, was then developed having in mind the process by which a class of algebras is associated with a given logic. Once this connection could be made, there was also interest in investigating the connection between various metalogical properties of the logic at hand and algebraic properties of the associated class of algebras. A recent overview of the area can be found in (Font *et al.*, 2003).

Although the Blok-Pigozzi approach appeared as a generalization of the Lindenbaum-Tarski construction, the algebraic notion of congruence remained as the main building block of AAL. So, at the light of the standard tools of AAL, the lack of a non-trivial congruence in a given logic invalidates the possibility of a meaningful algebraic study of the logic. This is precisely the case of many so-called non-truth-functional logics and, in particular, of da Costa's $\mathcal{C}$-systems, a hierarchy $\mathcal{C} = \{\mathcal{C}_n\}_{n\in\mathbb{N}}$ of logical systems introduced by da Costa in (da Costa, 1963; da Costa, 1974) for reasoning about formal inconsistency. As early as in (da Costa *et al.*, 1964), da Costa and Guillaume realized that the rule of replacement fails in each of these systems, due to the properties of the paraconsistent negation. This is the same as to say that logical equivalence is not a congruence relation in the algebra of formulas. Their non-Frégean nature hence rules out immediately the possibility of a Lindenbaum-Tarski algebraization of the logics. At this time, and despite this negative result, the problem of finding a Blok-Pigozzi algebraization of the $\mathcal{C}$-systems was still open. Mortensen finally gave a negative answer to this question in (Mortensen, 1980). He proved that the only congruence compatible with the set of theorems of $\mathcal{C}_1$ is trivial. As a consequence, every logic weaker than $\mathcal{C}_1$, which includes every member of the $\mathcal{C}_n$ hierarchy, fails to be algebraizable. In (Lewin *et al.*, 1991), Lewin, Mikenberg and Schwarze gave a small and easy proof that $\mathcal{C}_1$ is not algebraizable using the techniques of AAL. What is very interesting about this state of affairs, is that an algebraic flavored counterpart for these logics has been proposed and studied in the literature, as it was soon realized that, although not algebraizable, the $\mathcal{C}$-systems do indeed have some rich algebraic structure underneath, namely because their positive fragment is classical. Hence, da Costa himself advanced a proposal in (da Costa, 1966), which was then generalized by Carnielli and de Alcantara in (Carnielli *et al.*, 1984), and by Seoane and de Alcantara in (Seoane *et al.*, 1991): the class of so-called *da Costa algebras*. This class of algebraic structures was carefully studied and, in particular, a Stone-like representation theorem was proved, to the effect that every such structure is isomorphic to a *paraconsistent algebra of sets*. Still, it was clear that these algebraic structures could not be seen, in a strict sense, as algebras over the similarity type corresponding to the logical connectives. Other interesting semantics for the $\mathcal{C}$-systems have also been introduced, namely the bivaluation semantics of (Da Costa

et al., 1977), but the gap has remained open as none of these proposals could ever be explained by the standard theory of AAL.

However, the work in (Caleiro *et al.*, 2003) motivated a novel *behavioral* branch of AAL, as proposed and developed by Caleiro, Gonçalves and Martins in (Caleiro *et al.*, 2009). Intuitively, while AAL is traditionally centered around the notion of congruence, the behavioral approach to AAL is centered around the weaker notion of behavioral equivalence. Behavioral equivalence has its roots in computer science, namely in the field of algebraic specifications with applications to knowledge representation, software engineering, and object-oriented programming, where it is often necessary to reason about data which cannot be directly accessed. In fact, complex systems constitute a challenge for traditional algebraic methods, since they very often provide mechanisms to encapsulate internal data. In such situations, it is perfectly possible that one cannot distinguish between two different values if these two values provide exactly the same results for all available ways of observing and experimenting with them. In this case, we say that the two values are behaviorally equivalent (Reichel, 1985). Behavioral reasoning in equational logic has been consistently developed, see for instance (Goguen *et al.*, 2000; Rosu, 2004). In technical terms, the data is split into two categories: *visible data* which can be directly accessed, and *hidden data* that can only be accessed indirectly by analyzing the visible output of operations on it, suitably called *experiments*. Since one cannot access the hidden data, it is not possible to reason directly about the equality of two hidden values. Hence, unsorted equational logic is replaced by many-sorted behavioral equational logic (sometimes called hidden equational logic) based on the notion of behavioral equivalence, given a set of available experiments Γ. This restriction induces the notion of Γ-behavioral equivalence, where Γ is a subset of the set of original operations. It can be shown that the Γ-behavioral equivalence is the largest Γ-congruence whose visible part is the identity relation. Notably, the possibility of having a restricted set of experiments also accommodates the existence of non-congruent operations (Rosu, 2004). This feature of behavioral logic plays a fundamental role in our development of behavioral AAL, since it allows us to cope also with non-truth-functionality. In a nutshell, the behavioral approach to AAL introduced in (Caleiro *et al.*, 2009) extends the standard theory of AAL by simply using many-sorted behavioral logic in the role traditionally played by (unsorted) equational logic. For the sake of readability, we took the decision to avoid too many technicalities about behavioral equivalence, whenever possible, so that they do not interfere with the correct understanding of the key ideas we want to transmit.

Using behavioral tools and techniques in AAL, it is possible to prove that the whole hierarchy of da Costa's $\mathcal{C}$-systems is behaviorally algebraizable, as shown already in (Caleiro *et al.*, 2009; Gonçalves, 2008). Still, the task of studying the (behavioral) algebraization of a logic is not concluded when one manages to prove that the logic is (behaviorally) algebraizable. It is perhaps even more important to analyze the consequences of this fact, to study the properties of the algebraic counterpart of the logic, and to understand, ultimately, how they can contribute to a better understanding of the logic itself. This is precisely the purpose of the present paper. As we will

show, by studying the algebraic counterpart of da Costa's $\mathcal{C}$-systems, their connection with da Costa algebras, as well as with their bivaluation semantics, finally emerges naturally in this behavioral algebraic setting. The main contribution of the paper is precisely to reveal this connection [1]. For simplicity, we have focused most of our exposition on the logic $\mathcal{C}_1$, but it should be clear that the results transfer smoothly to the whole hierarchy of $\mathcal{C}$-systems.

The paper is organized as follows. In Section 2 we recall da Costa's $\mathcal{C}$-systems and some of their properties, including their bivaluation semantics and the class of da Costa structures. Special attention is dedicated to recalling why they cannot be algebraized using the standard tools of AAL. The behavioral algebraization of $\mathcal{C}_1$ will be overviewed in Section 3, including the presentation of the algebraic counterpart for $\mathcal{C}_1$ in this behavioral setting. Section 4 contains the main contributions of the paper, namely the study of the connection between the behavioral algebraization of $\mathcal{C}_1$ and the class of da Costa algebras. As a by-product, also the connection with its bivaluation semantics is explained. Finally, in Section 5, we draw some conclusions and point to some topics of further research.

2. The $\mathcal{C}$-systems

Let us start by recalling the hierarchy $\{\mathcal{C}_n\}_{n\in I\!N}$, and some of its properties. All the logic systems share a common language, $L_{\mathcal{C}}$, obtained by applying the primitive connectives $\neg$, $\wedge$, $\vee$, $\Rightarrow$ (respecting the usual arities) to a countable set P of propositional variables. Given a formula $\varphi \in L_{\mathcal{C}}$ we will sometimes write $\varphi(p_1, \ldots, p_n)$ to emphasize the fact that the variables of φ are all among $p_1, \ldots, p_n$. We will also use $\xi, \xi_1, \xi_2, \ldots$ as metavariables ranging over formulas.

Given $\varphi \in L_{\mathcal{C}}$, let $\varphi^\circ \triangleq \neg(\varphi \wedge \neg\varphi)$, and $\varphi^n \triangleq \varphi^{\overbrace{\circ\cdots\circ}^{n \text{ times}}}$, where the symbol $\circ$ appears $n \geq 1$ times, as well as $\varphi^{(n)} \triangleq (\varphi^1 \wedge \ldots \wedge \varphi^n)$. Indeed, since the logics are paraconsistent, formulas like φ°, φ^n and $\varphi^{(n)}$ are typically not theorems, and they can be used to express consistency within the $\mathcal{C}$-systems.

For each $n \in I\!N$, a Hilbert-style axiomatization of $\mathcal{C}_n$ can be obtained by using the fixed axioms (i-x)

1. An early version of this work was the subject of a contributed talk at the Fourth World Congress of Paraconsistency, held in Melbourne, Australia, in July 2008, whose short abstract appears in (Caleiro *et al.*, 2008a).

(i) $\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1)$
(ii) $(\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow (\xi_1 \Rightarrow \xi_3))$
(iii) $(\xi_1 \wedge \xi_2) \Rightarrow \xi_1$
(iv) $(\xi_1 \wedge \xi_2) \Rightarrow \xi_2$
(v) $\xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2))$
(vi) $\xi_1 \Rightarrow (\xi_1 \vee \xi_2)$
(vii) $\xi_2 \Rightarrow (\xi_1 \vee \xi_2)$
(viii) $(\xi_1 \Rightarrow \xi_3) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow ((\xi_1 \vee \xi_2) \Rightarrow \xi_3))$
(ix) $\xi_1 \vee \neg\xi_1$
(x) $\neg\neg\xi_1 \Rightarrow \xi_1$

plus the parametric axioms (xi$_n$-xiv$_n$)

(xi$_n$) $\xi_1^{(n)} \Rightarrow (\xi_1 \Rightarrow (\neg\xi_1 \Rightarrow \xi_2))$
(xii$_n$) $(\xi_1^{(n)} \wedge \xi_2^{(n)}) \Rightarrow (\xi_1 \wedge \xi_2)^{(n)}$
(xiii$_n$) $(\xi_1^{(n)} \wedge \xi_2^{(n)}) \Rightarrow (\xi_1 \vee \xi_2)^{(n)}$
(xiv$_n$) $(\xi_1^{(n)} \wedge \xi_2^{(n)}) \Rightarrow (\xi_1 \Rightarrow \xi_2)^{(n)}$

as well (MP) as the unique rule of inference

(MP) $\frac{\xi_1 \quad \xi_1 \Rightarrow \xi_2}{\xi_2}$.

We will denote the resulting consequence relation of each $\mathcal{C}_n$ by $\vdash_n$. In the case of $\mathcal{C}_1$, we will sometimes write also $\vdash_{\mathcal{C}_1}$. It is well-known that $\vdash_n \supsetneq \vdash_{n+1}$, for every n, which makes the hierarchy of $\mathcal{C}$-systems increasingly weak. It is clear that the deduction theorem holds for each $\mathcal{C}_n$.

The $\mathcal{C}$-systems are weaker than classical propositional logic (CPL), namely with respect to the properties of the paraconsistent negation $\neg$. Indeed, the $\mathcal{C}$-systems are *paraconsistent*, in the sense that a formula and its negation do not necessarily lead to deductive explosion. That is, in general in each $\mathcal{C}_n$, we have that $\varphi, \neg\varphi \nvdash_n \psi$. Still, in $\mathcal{C}_n$, $\varphi^{(n)}$ can be used to assert the consistency of a formula φ. Indeed, it is the case that $\varphi^{(n)}, \varphi, \neg\varphi \vdash_n \psi$, that is, deductive explosion reappears in the presence of consistency, as explained by axiom (xi$_n$). The parametric axioms (xii$_n$-xiv$_n$) above explain precisely how this notion of consistency propagates along the other logical connectives. In these terms, there is a strong connection between the consequence relation of CPL and those of the $\mathcal{C}$-systems. To explain this, let $\Phi \cup \{\psi\} \subseteq L_{\mathcal{C}}$, and V be the set of propositional variables that occur in $\Phi \cup \{\psi\}$. For any n, if we denote by $V^{(n)}$ the set $\{p^{(n)} : p \in V\}$, then we have that $\Phi \vdash_{\text{CPL}} \psi$ if and only if $\Phi, V^{(n)} \vdash_n \psi$. Moreover, one can take advantage of the way consistency can be expressed to define in each $\mathcal{C}_n$ a classical negation connective $\sim^n$. Indeed, it suffices to let $\sim^n\varphi \triangleq (\varphi^{(n)} \wedge \neg\varphi)$. The fact that $\sim^n$ indeed behaves like a classical negation is an essential feature of these logics. For example, we have that $(\varphi \vee \sim^n \varphi)$ and $(\sim^n (\varphi \wedge \sim^n \varphi))$ are both theorems of $\mathcal{C}_n$. Clearly one can also define *verum* and *falsum* by $\mathbf{t} \triangleq (\varphi \Rightarrow \varphi)$ and $\mathbf{f}^n \triangleq \sim^n\mathbf{t}$, respectively, where φ is a fixed but arbitrary formula of $L_{\mathcal{C}}$. For simplicity, in the case of $\mathcal{C}_1$, we will simply write $\sim$ and $\mathbf{f}$, instead

of $\sim^1$ and $\mathbf{f^1}$. In each $\mathcal{C}_n$, as usual, we can also define bi-implication as an abbreviation $\varphi \Leftrightarrow \psi \triangleq ((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi))$.

2.1. *Non-algebraizability*

We will proceed by giving a flavor of the traditional Blok-Pigozzi theory of algebraization of logics and making some consideration on why the $\mathcal{C}$-systems, being apparently well-behaved logics, fail to be algebraizable. Despite of their innocent aspect and the fact that they satisfy a lot of classical properties, each $\mathcal{C}_n$ is non-selfextensional. In general, it may happen that $\varphi \dashv\vdash_n \psi$ but $\neg\varphi \not\dashv\vdash_n \neg\psi$, where $\varphi \dashv\vdash_n \psi$ means that both $\varphi \vdash_n \psi$ and $\psi \vdash_n \varphi$ hold. This phenomenon, known as the lack of the intersubstitutivity of provable equivalents (IpE), has as an immediate consequence the fact that logical equivalence is not a congruence of the free algebra of formulas. This fact cuts short the possibility of a Lindenbaum-Tarski algebraization of the logics. Nevertheless, this fact alone does not put aside the possibility of an algebraization in the lines of the general theory of AAL. This task amounts to finding some nontrivial congruence on the algebra of formulas that takes into account the consequence relations of the $\mathcal{C}_n$. With respect to this last property, the minimum that one can impose on any such congruence is that it does not identify theorems with non-theorems. In this case we say that the congruence is compatible with the set of theorems. However, this is impossible as was first shown in (Mortensen, 1980), and then in (Lewin *et al.*, 1991). More than the mere impossibility of a Blok-Pigozzi algebraization of the $\mathcal{C}$-systems, both of these works made clear that the paraconsistent negation connective $\neg$ is the responsible for this failure. Mortensen's proof of this fact is long and hard, also due to the fact that, at that time, there was no general definition nor theory of algebraizable logics. However, the proof of Lewin, Mikenberg and Schwarze is short and enlightening. We briefly recall it below, since its main argument uses the powerful tools of AAL in a very interesting way. We should make clear that, in both cases, only the case of $\mathcal{C}_1$ was analyzed. For the sake of brevity, we shall do the same here. A straightforward result of AAL shows that, since $\mathcal{C}_1$ can be seen as an axiomatic extension of every other $\mathcal{C}_n$ with $n > 1$, the non-algebraizability of the former implies the non-algebraizability of the latter. It is absolutely straightforward, in any case, to generalize the arguments used to the whole $\{\mathcal{C}_n\}_{n\in I\!N}$ hierarchy.

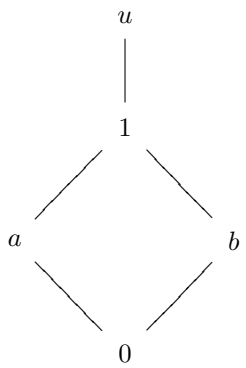
One of the important theorems of Blok and Pigozzi's seminal paper on AAL (Blok *et al.*, 1989), is the one that states that if a logic $\mathcal{L}$ is algebraizable and its equivalent algebraic semantics is the class of algebras K then, for every algebra $\mathbf{C}$ of the same similarity type, the Leibniz operator $\Omega_{\mathbf{C}}$ (mapping each $\mathcal{L}$-filter of $\mathbf{C}$ to the largest compatible K-congruence of $\mathbf{C}$) is an isomorphism. Note that a K-congruence of $\mathbf{C}$ is simply a congruence of $\mathbf{C}$ such that the corresponding reduced algebra is in the class K. So, in order to show that a given logic is not algebraizable, it suffices to present an algebra $\mathbf{C}$ such that $\Omega_{\mathbf{C}}$ is not an isomorphism. In (Lewin *et al.*, 1991),

the authors proved the non-algebraizability of $\mathcal{C}_1$ precisely by exhibiting a particular algebra where the Leibniz operator is not injective, namely

$$\mathbf{C} = \langle C, \mathbf{t}_{\mathbf{C}}, \mathbf{f}_{\mathbf{C}}, \wedge_{\mathbf{C}}, \vee_{\mathbf{C}}, \Rightarrow_{\mathbf{C}}, \neg_{\mathbf{C}} \rangle$$

such that

– $C = \{0, a, b, 1, u\}$;
– $\mathbf{t}_{\mathbf{C}} = 1$ and $\mathbf{f}_{\mathbf{C}} = 0$;
– $\langle C, \wedge_{\mathbf{C}}, \vee_{\mathbf{C}} \rangle$ is a lattice as depicted below;



– the operations $\Rightarrow_{\mathbf{C}}$ and $\neg_{\mathbf{C}}$ are defined by the following tables.

$\Rightarrow_{\mathbf{C}}$	u	1	a	b	0
u	u	u	a	b	0
1	u	1	a	b	0
a	u	1	1	b	b
b	u	1	a	1	a
0	u	1	1	1	1

	$\neg_{\mathbf{C}}$
u	1
1	0
a	b
b	a
0	1

The key feature of $\mathbf{C}$ is that it is a simple algebra, that is, it has no non-trivial congruences. So, we have just the trivial congruences

– $\mathbf{\Delta} = \{\langle x, x \rangle : x \in C\}$; and
– $\nabla = C \times C$.

It is easy to verify, also, that the $\mathcal{C}_1$-filters of $\mathbf{C}$ are

– $D_1 = \{u, 1\}$;
– $D_2 = \{u, 1, a\}$;
– $D_3 = \{u, 1, b\}$; and
– $D_4 = \{u, 1, a, b, 0\}$.

Therefore, it is obvious that $\Omega_{\mathbf{C}}$ cannot be injective. In particular, we have that $\Omega_{\mathbf{C}}(D_2) = \Omega_{\mathbf{C}}(D_3) = \mathbf{\Delta}$.

2.2. *da Costa algebras*

Before these negative results, there was an effort to set up an algebraic semantics for $\mathcal{C}_1$. In (da Costa, 1966), da Costa proposed a class of structures, that he called $\mathcal{C}_1$-algebras, as a possible algebraic counterpart for $\mathcal{C}_1$. Later on, in (Carnielli *et al.*, 1984), Carnielli and de Alcantara refined da Costa's ideas and defined a class of structures that became known as *da Costa algebras*. This class of structures was studied in detail in (Carnielli *et al.*, 1984; Seoane *et al.*, 1991), and namely an important Stone-like representation theorem was proved about them. Let us recall the definition of this class of algebraic structures, adapted to the particular case of $\mathcal{C}_1$. By a *da Costa algebra* we mean a structure

$$\mathbf{U} = \langle U, 0, 1, \leq_{\mathbf{U}}, \wedge_{\mathbf{U}}, \vee_{\mathbf{U}}, \Rightarrow_{\mathbf{U}}, \neg_{\mathbf{U}} \rangle,$$

such that $0, 1 \in U$ and, for every $a, b, c \in U$, the following conditions hold

– $\leq_{\mathbf{U}} \subseteq U \times U$ is a quasi-order, that is, it satisfies

  Reflexivity: $a \leq_{\mathbf{U}} a$, and

  Transitivity: if $a \leq_{\mathbf{U}} b$ and $b \leq_{\mathbf{U}} c$ then $a \leq_{\mathbf{U}} c$;

– $(a \wedge_{\mathbf{U}} b) \leq_{\mathbf{U}} a$ and $(a \wedge_{\mathbf{U}} b) \leq_{\mathbf{U}} b$;

– $(a \wedge_{\mathbf{U}} a) \simeq_{\mathbf{U}} a$ and $(a \vee_{\mathbf{U}} a) \simeq_{\mathbf{U}} a$, where $a \simeq_{\mathbf{U}} b$ iff $a \leq_{\mathbf{U}} b$ and $b \leq_{\mathbf{U}} a$;

– $(a \wedge_{\mathbf{U}} (b \vee_{\mathbf{U}} c)) \simeq_{\mathbf{U}} ((a \wedge_{\mathbf{U}} b) \vee_{\mathbf{U}} (a \wedge_{\mathbf{U}} c))$;

– $a \leq_{\mathbf{U}} (a \vee_{\mathbf{U}} b)$ and $b \leq_{\mathbf{U}} (a \vee_{\mathbf{U}} b)$;

– if $a \leq_{\mathbf{U}} c$ and $b \leq_{\mathbf{U}} c$ then $(a \vee_{\mathbf{U}} b) \leq_{\mathbf{U}} c$;

– $(a \wedge_{\mathbf{U}} (a \Rightarrow_{\mathbf{U}} b)) \leq_{\mathbf{U}} b$;

– $(a \wedge_{\mathbf{U}} c) \leq_{\mathbf{U}} b$ then $c \leq_{\mathbf{U}} (a \Rightarrow_{\mathbf{U}} b)$;

– $0 \leq_{\mathbf{U}} a$ and $a \leq_{\mathbf{U}} 1$;

– $a^\circ \leq_{\mathbf{U}} (\neg_{\mathbf{U}} a)^\circ$, where $a^\circ \triangleq \neg_{\mathbf{U}}(a \wedge_{\mathbf{U}} \neg_{\mathbf{U}} a)$;

– $(a \vee_{\mathbf{U}} \neg_{\mathbf{U}} a) \simeq_{\mathbf{U}} 1$;

– $\neg_{\mathbf{U}}(\neg_{\mathbf{U}} x) \leq_{\mathbf{U}} x$;

– $a^\circ \leq_{\mathbf{U}} ((b \Rightarrow_{\mathbf{U}} a) \Rightarrow_{\mathbf{U}} ((b \Rightarrow_{\mathbf{U}} \neg_{\mathbf{U}} a) \Rightarrow_{\mathbf{U}} \neg_{\mathbf{U}} b))$;

– $(a^\circ \wedge_{\mathbf{U}} (\neg_{\mathbf{U}} a^\circ)) \simeq_{\mathbf{U}} 0$;

– $(a^\circ \wedge_{\mathbf{U}} b^\circ) \leq_{\mathbf{U}} (a \wedge_{\mathbf{U}} b)^\circ$;

– $(a^\circ \wedge_{\mathbf{U}} b^\circ) \leq_{\mathbf{U}} (a \vee_{\mathbf{U}} b)^\circ$;

– $(a^\circ \wedge_{\mathbf{U}} b^\circ) \leq_{\mathbf{U}} (a \Rightarrow_{\mathbf{U}} b)^\circ$.

Expectedly, $\Phi \vdash_{\mathcal{C}_1} \psi$ iff for every da Costa algebra $\mathbf{U}$ and assignment $h : P \to U$ we have that $\psi^h_{\mathbf{U}} \simeq_{\mathbf{U}} 1$ whenever $\varphi^h_{\mathbf{U}} \simeq_{\mathbf{U}} 1$ for every $\varphi \in \Phi$.

Although there is a strong conceptual connection between $\mathcal{C}_1$ and da Costa algebras, the negative results discussed in Subsection 2.1 make it impossible to explain this connection at the light of the traditional theory of AAL. Clearly, the algebraic

structures in this class cannot even be seen as algebras over the similarity type corresponding to the logical connectives of $\mathcal{C}_1$.

2.3. *Bivaluation semantics*

The first major semantical analysis of the logics $\mathcal{C}_n$ was developed in (Da Costa *et al.*, 1977) by da Costa and Alves, where they proposed a non-truth-functional bivaluation semantics for each logic in the $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ hierarchy. Namely, a bivaluation for $\mathcal{C}_1$ is a function $\nu : L_{\mathcal{C}} \to \{0,1\}$ that satisfies the following conditions

– $\nu(\varphi_1 \wedge \varphi_2) = 1$ iff $\nu(\varphi_1) = 1$ and $\nu(\varphi_2) = 1$;
– $\nu(\varphi_1 \vee \varphi_2) = 1$ iff $\nu(\varphi_1) = 1$ or $\nu(\varphi_2) = 1$;
– $\nu(\varphi_1 \Rightarrow \varphi_2) = 1$ iff $\nu(\varphi_1) = 0$ or $\nu(\varphi_2) = 1$;
– if $\nu(\varphi_2^\circ) = \nu(\varphi_1 \Rightarrow \varphi_2) = \nu(\varphi_1 \Rightarrow \neg\varphi_2) = 1$ then $\nu(\varphi_1) = 0$;
– if $\nu(\varphi_1^\circ) = \nu(\varphi_2^\circ) = 1$ then $\nu((\varphi_1 * \varphi_2)^\circ) = 1$ where $* \in \{\wedge, \vee, \Rightarrow\}$;
– if $\nu(\varphi) = 0$ then $\nu(\neg\varphi) = 1$;
– if $\nu(\neg\neg\varphi) = 1$ then $\nu(\varphi) = 1$.

As shown in (Da Costa *et al.*, 1977), $\Phi \vdash_{\mathcal{C}_1} \psi$ iff $\nu(\varphi) = 1$ for each $\varphi \in \Phi$ implies $\nu(\psi) = 1$, for every $\mathcal{C}_1$ bivaluation ν.

Valuation semantics was advocated in (da Costa *et al.*, 1994) as a very general kind of semantics. It is indeed very general, and very useful for logics that do not admit a truth-functional semantics, like $\mathcal{C}_1$. Although the bivaluation semantics for $\mathcal{C}_1$ appeared much earlier than all the non-algebraizability concerns, it will be useful for us in the sequel. In any case, it certainly sheds some light on the subtleties of the logic.

3. Behavioral algebraization

Recall that, when setting-up an algebraic semantics for a truth-functional logic, we endow it with models that are algebras over the connectives of the logic, and we evaluate formulas homomorphically. This approach, using truth-functional logical matrices, does not work when the logic is not truth-functional. A change of paradigm in (Caleiro *et al.*, 2003), led to proposing an alternative algebraic semantics for $\mathcal{C}_1$. The idea was to associate to $\mathcal{C}_1$ not a class of algebras over the signature of $\mathcal{C}_1$, but rather a class of algebras over a two-sorted signature of formulas and truth-values, where the valuation map is an operation between the two sorts. The work presented here could not be fully systematized in this early setting, namely due to the essential role played by behavioral reasoning. However, it is only fair to say that it inspired a novel behavioral approach to AAL, as introduced in (Caleiro *et al.*, 2009), whose main aim was extending the range of applicability of the usual tools of AAL towards providing a meaningful algebraic counterpart also to logics that, as $\mathcal{C}_1$, include non-truth-

functional connectives and which are not algebraizable according to the traditional theory. However, the key idea goes well beyond many-sortedness, as in the behavioral approach to AAL we substitute the role played by *unsorted equational logic* (*Eqn*) by a suitable *many-sorted behavioral equational logic* (*BEqn*). As *Eqn*, *BEqn* is a logic for reasoning about equations. In *BEqn*, however, equality is only defined relatively to a prescribed set of possible experiments, as explained above.

For those who might not be familiar with the terminology, the example of first-order logic can be helpful. There we have the logical language divided in the sort of terms and the sort of formulas. Function symbols can be viewed as operations on terms, whereas predicates can be viewed as operations that transform terms into formulas. The connectives and quantifiers are simply operations on formulas. The usual interpretation structures can be viewed as algebras over this two-sorted signature, where we have a set to interpret terms, the domain of interpretation, and a set to interpret formulas, typically the set $\{0, 1\}$. Function symbols are interpreted as functions on the domain, and predicates, usually interpreted as relations, can be interpreted as functions from the domain to $\{0, 1\}$. Connectives and quantifiers bear their usual interpretations as functions over $\{0, 1\}$.

The distinctive feature of *BEqn* is that the sorts are split in two disjoint sets, of *visible* and *hidden* sorts, and that the operations are also split in two disjoint sets, of *behavioral* and *non-behavioral* (or *congruent* and *non-congruent*) operations. The behavioral operations generate the *experiments*, which are visible terms with a distinguished hidden variable. The experiments can be seen as the only way to access the hidden data by substituting the distinguished variable of the experiment by the hidden term to observe. It this way, they generate a *behavioral equivalence* on each algebra, by defining two elements to be behaviorally equivalent if and only if they cannot be distinguished by any experiment evaluated in that algebra. Obviously, the behavioral equivalence relation is preserved by the behavioral operations in any algebra, but it may not be preserved by the non-congruent operations in some algebras. An algebra $\mathbf{A}$ behaviorally satisfies an equation $t_1 \approx t_2$ if for every assignment h over $\mathbf{A}$, the value of t_1 is behaviorally equivalent to the value of t_2. Hence, we can say that *BEqn* is defined as *Eqn* but where usual equational satisfaction is replaced by behavioral satisfaction. We will present these notions rigorously below, when applied to the particular case of $\mathcal{C}_1$.

The precise definition of *behaviorally algebraizable logic* is very technical and it is not our objective to get into these technical details here. The interested reader can find all the details in (Caleiro *et al.*, 2009; Gonçalves, 2008). We can say, informally, that a propositional logic is behaviorally algebraizable if it is equivalent to the behavioral equational theory of a class of two-sorted algebras. To allow for non-congruent connectives, the sort of formulas must be a hidden sort, so that one is forced to reason behaviorally about formulas. This can be achieved by considering behavioral logic over an extended signature.

In the concrete case of the logic $\mathcal{C}_1$, we should consider a two-sorted signature

$$\Sigma_{\mathcal{C}_1} = \langle \{\phi, v\}, \{\wedge, \vee, \Rightarrow, \sim, \neg, obs\} \rangle$$

where:

– ϕ denotes the sort of formulas of the logic, v denotes the sort of truth-values;

– the connectives have their usual types on sort ϕ, and the *observation* operation [2] generates elements of sort v, as shown below.

$$\begin{array}{llllll} \wedge: & \phi^2 \to \phi & \vee: & \phi^2 \to \phi & \Rightarrow: & \phi^2 \to \phi \\ \sim: & \phi \to \phi & \neg: & \phi \to \phi & obs: & \phi \to v \end{array}$$

Intuitively, we are just extending the propositional signature with a new sort v for the observations that we can perform on formulas using the unary operation obs. In any case, the formulas of $L_{\mathcal{C}}$ can be easily identified with the terms of sort ϕ. For simplicity, we have also included the classical negation $\sim$ as primitive. The choice of v as the name for the new sort is clear, as it consists of the only visible sort of the extended signature. As congruent operations we will take all operations on the hidden sort ϕ except paraconsistent negation, that is,

$$\Gamma_{\mathcal{C}_1} = \{\wedge, \vee, \Rightarrow, \sim\}.$$

In this way, the experiments are all of the form $obs(\gamma)$, where $\gamma = \gamma(x, x_1, \ldots, x_n)$ is a formula built just with the classical connectives of $\mathcal{C}_1$, with a distinguished variable x (where the experiment takes place), and parametric variables $x_1, \ldots, x_n$. Along with the appropriate congruence conditions on the observations, this will allow us to study the structure of the resulting (classical) visible values. Note that one can also observe formulas that include the paraconsistent negation by substituting them in x. However, we will have no way of relating, for instance, $obs(\psi)$ and $obs(\neg\psi)$, as $obs(\neg x)$ is not a valid experiment.

An algebra over $\Sigma_{\mathcal{C}_1}$ is a tuple $\mathbf{A} = \langle A_\phi, A_v, \wedge_{\mathbf{A}}, \vee_{\mathbf{A}}, \Rightarrow_{\mathbf{A}}, \sim_{\mathbf{A}}, \neg_{\mathbf{A}}, obs_{\mathbf{A}} \rangle$, where A_ϕ is a set to interpret formulas, A_v a set to interpret the observation of formulas, and the remainder are the interpretations of the operations. As usual, we say that a $\Sigma_{\mathcal{C}_1}$-algebra $\mathbf{A}$ satisfies an equation $\varphi \approx \psi$, which is denoted by $\mathbf{A} \Vdash \varphi \approx \psi$, if for every assignment h of values to the variables we have that $\mathbf{A}, h \Vdash \varphi \approx \psi$, that is, $\varphi^h_{\mathbf{A}} = \psi^h_{\mathbf{A}}$. Given a class $\mathbb{K}$ of $\Sigma_{\mathcal{C}_1}$-algebras and a set Θ of equations we say that $\Theta \models_{\mathbb{K}} \varphi \approx \psi$ if for every algebra $\mathbf{A} \in \mathbb{K}$ and for every assignment h on $\mathbf{A}$ we have that $\mathbf{A}, h \Vdash \varphi \approx \psi$ whenever $\mathbf{A}, h \Vdash \theta_1 \approx \theta_2$ for every $\theta_1 \approx \theta_2 \in \Theta$. We say that $\mathbf{A}$ satisfies a conditional-equation $(\varphi_1 \approx \psi_1 \,\&\, \ldots \,\&\, \varphi_n \approx \psi_n) \to \varphi \approx \psi$, which is denoted by $\mathbf{A} \Vdash (\varphi_1 \approx \psi_1 \,\&\, \ldots \,\&\, \varphi_n \approx \psi_n) \to \varphi \approx \psi$, whenever it holds that $\{\varphi_1 \approx \psi_1, \ldots, \varphi_n \approx \psi_n\} \models_{\{\mathbf{A}\}} \varphi \approx \psi$.

2. In this paper, in order to guarantee that it would not be confused with the $\_^\circ$ notation used in $\mathcal{C}_1$, we named the observation operation *obs* instead of simply *o* in the notation of (Caleiro *et al.*, 2009; Gonçalves, 2008).

To define behavioral satisfaction in an algebra $\mathbf{A}$, we first need to introduce the notion of *behavioral equivalence* (modulo $\Gamma_{\mathcal{C}_1}$). We say that $a, b \in A_\phi$ are *behaviorally equivalent* in $\mathbf{A}$, denoted by $a \equiv^{\mathcal{C}_1}_{\mathbf{A}} b$, if for every experiment $obs(\gamma(x, x_1, \ldots, x_n))$ and every assignment h to the variables, we have that $obs(\gamma(a, x_1, \ldots, x_n))^h_{\mathbf{A}} = obs(\gamma(b, x_1, \ldots, x_n))^h_{\mathbf{A}}$. One can now understand why behavioral equivalence amounts to indistinguishability by means of experiments.

We say that $\mathbf{A}$ behaviorally satisfies an equation $\varphi \approx \psi$, which we denote by writing $\mathbf{A} \Vdash^{\Gamma_{\mathcal{C}_1}} \varphi \approx \psi$, if for every assignment h of values to the variables we have that $\mathbf{A}, h \Vdash^{\Gamma_{\mathcal{C}_1}} \varphi \approx \psi$, that is, $\varphi^h_{\mathbf{A}} \equiv^{\mathcal{C}_1}_{\mathbf{A}} \psi^h_{\mathbf{A}}$. Given a class $\mathbb{K}$ of $\Sigma_{\mathcal{C}_1}$-algebras and a set Θ of equations we say that $\Theta \models^{\Gamma_{\mathcal{C}_1}}_{\mathbb{K}} \varphi \approx \psi$ if for every algebra $\mathbf{A} \in \mathbb{K}$ and for every assignment h on $\mathbf{A}$ we have that $\mathbf{A}, h \Vdash^{\Gamma_{\mathcal{C}_1}} \varphi \approx \psi$ whenever $\mathbf{A}, h \Vdash^{\Gamma_{\mathcal{C}_1}} \theta_1 \approx \theta_2$ for every $\theta_1 \approx \theta_2 \in \Theta$. Analogously, we say that $\mathbf{A}$ behaviorally satisfies a conditional-equation, which is denoted by $\mathbf{A} \Vdash^{\Gamma_{\mathcal{C}_1}} (\varphi_1 \approx \psi_1 \,\&\, \ldots \,\&\, \varphi_n \approx \psi_n) \rightarrow \varphi \approx \psi$, if $\{\varphi_1 \approx \psi_1 \,\&\, \ldots \,\&\, \varphi_n \approx \psi_n\} \models^{\Gamma_{\mathcal{C}_1}}_{\{\mathbf{A}\}} \varphi \approx \psi$. The reader should contrast these definitions with the usual ones of satisfaction of a (conditional-)equation given above at the light of the notion of behavioral equivalence, as the differences will play a key role in the subsequent development.

3.1. $\mathbb{K}_{\mathcal{C}_1}$-*algebras*

It is not our intention to dwell here on the technical details of the behavioral algebraization of $\mathcal{C}_1$. What is important to retain, for our purposes here, is that it was proved in (Caleiro *et al.*, 2009) that $\mathcal{C}_1$ is behaviorally algebraizable and therefore a class $\mathbb{K}_{\mathcal{C}_1}$ of two-sorted $\Sigma_{\mathcal{C}_1}$-algebras can be canonically associated to it. In the remainder of this section we will present $\mathbb{K}_{\mathcal{C}_1}$ and explore some of its properties. Further details can be found in (Gonçalves, 2008). The essential idea underlying the definition of $\mathbb{K}_{\mathcal{C}_1}$ is the fact that the bi-implication of the logic does not define a congruence over the whole language, but it internalizes precisely the $\Gamma_{\mathcal{C}_1}$-behavioral equivalence. The class $\mathbb{K}_{\mathcal{C}_1}$ consists of all the $\Sigma_{\mathcal{C}_1}$-algebras $\mathbf{A}$, with $obs_{\mathbf{A}}$ a surjective function, that behaviorally satisfy the following hidden equations

(1) $\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1) \approx \mathbf{t}$;
(2) $(\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow (\xi_1 \Rightarrow \xi_3)) \approx \mathbf{t}$;
(3) $(\xi_1 \wedge \xi_2) \Rightarrow \xi_1 \approx \mathbf{t}$;
(4) $(\xi_1 \wedge \xi_2) \Rightarrow \xi_2 \approx \mathbf{t}$;
(5) $\xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2)) \approx \mathbf{t}$;
(6) $\xi_1 \Rightarrow (\xi_1 \vee \xi_2) \approx \mathbf{t}$;
(7) $\xi_2 \Rightarrow (\xi_1 \vee \xi_2) \approx \mathbf{t}$;
(8) $(\xi_1 \Rightarrow \xi_3) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow ((\xi_1 \vee \xi_2) \Rightarrow \xi_3)) \approx \mathbf{t}$;
(9) $\xi_1 \vee \neg\xi_1 \approx \mathbf{t}$;
(10) $\neg\neg\xi_1 \Rightarrow \xi_1 \approx \mathbf{t}$;

(11) $\xi_1^\circ \Rightarrow (\xi_1 \Rightarrow (\neg\xi_1 \Rightarrow \xi_2)) \approx \mathbf{t}$;
(12) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \wedge \xi_2)^\circ \approx \mathbf{t}$;
(13) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \vee \xi_2)^\circ \approx \mathbf{t}$;
(14) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \Rightarrow \xi_2)^\circ \approx \mathbf{t}$,

and behaviorally satisfy, also, the following hidden conditional-equations

(15) $(\xi_1 \approx \mathbf{t} \ \& \ \xi_1 \Rightarrow \xi_2 \approx \mathbf{t}) \ \rightarrow \ \xi_2 \approx \mathbf{t}$;
(16) $(\xi_1 \Rightarrow \xi_2 \approx \mathbf{t} \ \& \ \xi_2 \Rightarrow \xi_1 \approx \mathbf{t}) \ \rightarrow \ \xi_1 \approx \xi_2$,

and further satisfy the following visible conditional-equations

(17) $(obs(\xi_1) \approx obs(\xi_2)) \rightarrow obs(\sim \xi_1) \approx obs(\sim \xi_2)$;
(18) $(obs(\xi_1) \approx obs(\xi_2)$ & $obs(\xi_3) \approx obs(\xi_4)) \rightarrow obs(\xi_1 \vee \xi_3) \approx obs(\xi_2 \vee \xi_4)$;
(19) $(obs(\xi_1) \approx obs(\xi_2)$ & $obs(\xi_3) \approx obs(\xi_4)) \rightarrow obs(\xi_1 \wedge \xi_3) \approx obs(\xi_2 \wedge \xi_4)$;
(20) $(obs(\xi_1) \approx obs(\xi_2)$ & $obs(\xi_3) \approx obs(\xi_4)) \rightarrow obs(\xi_1 \Rightarrow \xi_3) \approx obs(\xi_2 \Rightarrow \xi_4)$.

Note that the way such a characterization of $\mathbb{K}_{\mathcal{C}_1}$ could be constructed from the axiomatization of $\mathcal{C}_1$, and vice-versa, is the result of applying to $\mathcal{C}_1$ a general result of AAL, which extends to the behavioral setting as shown in (Caleiro *et al.*, 2009). The axioms (i-xiv) of $\mathcal{C}_1$ directly give rise to conditions (1-14), and the (MP) rule gives rise to condition (15). The hidden conditional-equation (16) forces bi-implication $\Leftrightarrow$ to be interpreted as behavioral equivalence in every member of $\mathbb{K}_{\mathcal{C}_1}$. The visible equations (17-20) express the fact that the observation operation obs is well-behaved with respect to the congruent operations, that is, the classical connectives.

The following proposition states a fundamental property of the class $\mathbb{K}_{\mathcal{C}_1}$, stemming from the fact that it constitutes a behavioral algebraic semantics for $\mathcal{C}_1$.

Proposition 1. *Let $\Phi \cup \{\psi\} \subseteq L_{\mathcal{C}_1}$. Then the following conditions are equivalent:*

- $\Phi \vdash_{\mathcal{C}_1} \psi$;
- $\{\varphi \approx \mathbf{t} : \varphi \in \Phi\} \models^{\Gamma_{\mathcal{C}_1}}_{\mathbb{K}_{\mathcal{C}_1}} \psi \approx \mathbf{t}$; *and*
- $\{obs(\varphi) \approx \top : \varphi \in \Phi\} \models_{\mathbb{K}_{\mathcal{C}_1}} obs(\psi) \approx \top$.

We will briefly sketch the proof of this proposition. Additional details can be found in (Caleiro *et al.*, 2009; Gonçalves, 2008).

The equivalence between the first two conditions is a completeness result that closely relates the entailment of $\mathcal{C}_1$ with the behavioral equational consequence associated with $\mathbb{K}_{\mathcal{C}_1}$, as a consequence of the internalization of behavioral equivalence using the bi-implication $\Leftrightarrow$, at the light of condition (16). The fact that if $\Phi \vdash_{\mathcal{C}_1} \psi$ then $\{\varphi \approx \mathbf{t} : \varphi \in \Phi\} \models^{\Gamma_{\mathcal{C}_1}}_{\mathbb{K}_{\mathcal{C}_1}} \psi \approx \mathbf{t}$ is a direct consequence of the definition of $\mathbb{K}_{\mathcal{C}_1}$, namely of the fact that conditions (1-14) algebraically mirror all the axioms of $\mathcal{C}_1$, and condition (15) algebraically mirrors the only rule of $\mathcal{C}_1$. For the converse implication it is useful to take advantage of the completeness of $\mathcal{C}_1$ with respect to its bivaluation semantics. Indeed, every bivaluation for $\mathcal{C}_1$, as defined in subsection 2.3, has a repre-

sentative algebra in $\mathbb{K}_{\mathcal{C}_1}$. More precisely, given a bivaluation ν for $\mathcal{C}_1$ we can consider the algebra $\mathbf{A}_\nu$ that coincides with the free algebra of formulas when restricted to the sort ϕ, whose carrier of sort v is the set $\{0, 1\}$, and such that the operation obs is interpreted precisely as ν. It is straightforward to check that $A_\nu \in \mathbb{K}_{\mathcal{C}_1}$ for every bivaluation ν for $\mathcal{C}_1$.

The equivalence between the last two conditions is an important result since it transforms behavioral reasoning over formulas into equational reasoning over the observation of formulas. This equivalence is a direct consequence of the construction of $\mathbb{K}_{\mathcal{C}_1}$, namely with respect to the conditions (17-20). Recall that, roughly speaking, two elements are behaviorally equivalent if they cannot be distinguished by any experiment. The valid experiments are all of the form $obs(\gamma(x, x_1, \ldots, x_n))$ where γ is a formula build up with operations from $\Gamma_{\mathcal{C}_1}$. Hence, given that conditions (17-20) of the definition of $\mathbb{K}_{\mathcal{C}_1}$ guarantee that the operation obs is compatible with every element of $\Gamma_{\mathcal{C}_1}$, for proving that two elements are behaviorally equivalent it suffices to check whether the operation obs applied to both gives the same result.

As a particular case of the above proposition we have that $\vdash_{\mathcal{C}_1} \varphi$ iff $\models^{\Gamma_{\mathcal{C}_1}}_{\mathbb{K}_{\mathcal{C}_1}} \varphi \approx \mathbf{t}$ iff $\models_{\mathbb{K}_{\mathcal{C}_1}} obs(\varphi) \approx \top$.

Recall that we have not included in the signature $\Sigma_{\mathcal{C}_1}$ any operations on sort v. Nevertheless, it is interesting to note that given the set $\Gamma_{\mathcal{C}_1}$ of congruent operations, we can define by abbreviation some natural operations on the visible sort v. Namely,

– constant $\top \triangleq obs(\mathbf{t})$;
– constant $\bot \triangleq obs(\mathbf{f})$;
– binary meets $obs(x) \sqcap obs(y) \triangleq obs(x \wedge y)$;
– binary joins $obs(x) \sqcup obs(y) \triangleq obs(x \vee y)$;
– an implication $obs(x) \sqsupset obs(y) \triangleq obs(x \Rightarrow y)$;
– a bi-implication $obs(x) \equiv obs(y) \triangleq obs(x \Leftrightarrow y)$; and
– a unary complement $-obs(x) \triangleq obs(\sim x)$.

Since $obs_{\mathbf{A}}$ is required to be surjective, we have that $A_v = \{obs_{\mathbf{A}}(a) : a \in A_\phi\}$, for every $\mathbf{A} \in \mathbb{K}_{\mathcal{C}_1}$. This means that every value in A_v can be obtained as the observation of some formula (element of A_ϕ). This fact, together with the conditional-equations (17-20), gives us the guarantee that the interpretation of these operations is well-defined in every algebra $\mathbf{A} \in \mathbb{K}_{\mathcal{C}_1}$. Namely, given $\mathbf{A} \in \mathbb{K}_{\mathcal{C}_1}$ and $a_1, a_2, b_1, b_2 \in A_\phi$, if $obs_{\mathbf{A}}(a_1) = obs_{\mathbf{A}}(a_2)$ and $obs_{\mathbf{A}}(b_1) = obs_{\mathbf{A}}(b_2)$ then also $obs_{\mathbf{A}}(a_1) \sqcap_{\mathbf{A}} obs_{\mathbf{A}}(b_1) = obs_{\mathbf{A}}(a_2) \sqcap_{\mathbf{A}} obs_{\mathbf{A}}(b_2)$. Along this path, we can further unveil the intimate connection between $\mathcal{C}_1$ and CPL. This is a very important and useful result due to the strong and well-studied properties of Boolean algebras.

Proposition 2. *If* $\mathbf{A} \in \mathbb{K}_{\mathcal{C}_1}$ *then* $\mathbf{A}_v = \langle A_v, \top_{\mathbf{A}}, \bot_{\mathbf{A}}, \sqcap_{\mathbf{A}}, \sqcup_{\mathbf{A}}, \sqsupset_{\mathbf{A}}, -_{\mathbf{A}} \rangle$ *is a Boolean algebra.*

Finally, the next result states that $\mathcal{C}_1$ is indeed behaviorally algebraizable.

Theorem 3. *The logic $\mathcal{C}_1$ is $\Gamma_{\mathcal{C}_1}$-behaviorally algebraizable, with behaviorally equivalent algebraic semantics the class of $\mathbb{K}_{\mathcal{C}_1}$-algebras, with defining equation $\xi \approx \mathbf{t}$ and equivalence formula $\xi_1 \Leftrightarrow \xi_2$.*

We did not present the general definition of behaviorally algebraizable logic, but we can hint at what it means in this particular case along with sketching the proof of this theorem. As it turns out, the result amounts to showing the following properties:

(a) $\Phi \vdash_{\mathcal{C}_1} \psi$ iff $\{\varphi \approx \mathbf{t} : \varphi \in \Phi\} \models^{\Gamma_{\mathcal{C}_1}}_{\mathbb{K}_{\mathcal{C}_1}} \psi \approx \mathbf{t}$;

(b) $\{\varphi_i \approx \psi_i : i \in I\} \models^{\Gamma_{\mathcal{C}_1}}_{\mathbb{K}_{\mathcal{C}_1}} \varphi \approx \psi$ iff $\{\varphi_i \Leftrightarrow \psi_i : i \in I\} \vdash_{\mathcal{C}_1} \varphi \Leftrightarrow \psi$;

(c) $\xi \dashv\vdash_{\mathcal{C}_1} (\xi \Leftrightarrow \mathbf{t})$;

(d) $\xi_1 \approx \xi_2 \exists\models^{\Gamma_{\mathcal{C}_1}}_{\mathbb{K}_{\mathcal{C}_1}} (\xi_1 \Leftrightarrow \xi_2) \approx \mathbf{t}$.

We have intendedly presented the four conditions above to show the inherent symmetry of the definition, but it should be noted that they are not independent of each other. In fact, a smaller set of necessary and sufficient conditions can be presented. Namely, as in standard AAL, conditions (a) and (d) are jointly equivalent to conditions (b) and (c). Note that (a) corresponds precisely to the equivalence between the first two conditions of Proposition 1. Using an argument similar to that of Proposition 1, it can be shown that proving condition (d) is equivalent to checking that $obs(\xi_1) \approx obs(\xi_2) \models_{\mathbb{K}_{\mathcal{C}_1}} obs(\xi_1) \equiv obs(\xi_2) \approx \top$ and $obs(\xi_1) \equiv obs(\xi_2) \approx \top \models_{\mathbb{K}_{\mathcal{C}_1}} obs(\xi_1) \approx obs(\xi_2)$, and both conditions follow from the fact that $\mathbf{A}_v$ is a Boolean algebra, for every $\mathbf{A} \in \mathbb{K}_{\mathcal{C}_1}$. Again, details can be found in (Caleiro *et al.*, 2009; Gonçalves, 2008).

3.2. *Lewin, Mikenberg and Schwarze's argument no longer applies*

Our aim is now to reanalyze the proof of standard non-algebraizability of $\mathcal{C}_1$ discussed above, but now at the light of its behavioral algebraization. We want to explain why the argument used in the proof in (Lewin *et al.*, 1991) no longer applies in the behavioral setting. The general result of AAL used in Lewin, Mikenberg and Schwarze's proof has a corresponding behavioral version, as shown in (Caleiro *et al.*, 2009; Gonçalves, 2008). Namely, it states that if a logic $\mathcal{L}$ is Γ-behaviorally algebraizable and its behaviorally equivalent algebraic semantics is the class of algebras $\mathbb{K}$ then, for every algebra $\mathbf{C}$, the behavioral Leibniz operator $\Omega^{\Gamma}_{\mathbf{C}}$ (mapping each $\mathcal{L}$-filter of $\mathbf{C}$ to the largest compatible $\mathbb{K}$-Γ-congruence of $\mathbf{C}$) is an isomorphism. Note now, that a $\mathbb{K}$-Γ-congruence of $\mathbf{C}$ is a Γ-congruence such that the corresponding two-sorted reduced algebra is in $\mathbb{K}$. The two-sorted reduced algebra is a technical artifact that allows one to simulate a quotient of the algebra using a Γ-congruence, which in general is not really a congruence. We shall not go into the fine details of these notions here, but we give an hint at the related constructions below.

Let $\mathbf{C}$ be the algebra used in the proof of (Lewin *et al.*, 1991), that we recall from Subsection 2.1. As we have seen, there are four $\mathcal{C}_1$-filters of $\mathbf{C}$, namely D_1, D_2, D_3,

and D_4. Although $\mathbf{C}$ only has the trivial congruences Δ and ∇, it does have non-trivial $\Gamma_{\mathcal{C}_1}$-congruences (where, of course, the paraconsistent negation $\neg$ is disregarded). In fact, it is easy to see that if we restrict the operations of the algebra $\mathbf{C}$ to the elements $\{a, b, 0, 1\}$ we get a Boolean algebra. It suffices to use the abbreviation of classical negation, in order to obtain the interpretation shown in the table below.

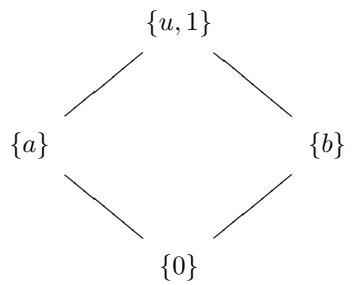
	$\sim_{\mathbf{C}}$
u	0
1	0
a	b
b	a
0	1

Therefore, $\mathbf{C}$ has four distinct $\mathbb{K}_{\mathcal{C}_1}$-$\Gamma_{\mathcal{C}_1}$-congruences, whose corresponding sets of equivalence classes are

- $\theta_1 = \{\{u, 1\}, \{a\}, \{b\}, \{0\}\}$;
- $\theta_2 = \{\{u, 1, a\}, \{b, 0\}\}$;
- $\theta_3 = \{\{u, 1, b\}, \{a, 0\}\}$; and
- $\theta_4 = \{\{u, 1, a, b, 0\}\}$.

It is easy to verify that the $\Gamma_{\mathcal{C}_1}$-behavioral Leibniz operator over $\mathbf{C}$ is such that $\Omega^{\Gamma_{\mathcal{C}_1}}_{\mathbf{C}}(D_i) = \theta_i$ for $i = 1, \ldots, 4$. Therefore, $\Omega^{\Gamma_{\mathcal{C}_1}}_{\mathbf{C}}$ is an isomorphism.

In order to shed some light on the construction of the corresponding two-sorted algebras, let us consider the $\Gamma_{\mathcal{C}_1}$-congruence θ_1. Then, the two-sorted reduced algebra $\mathbf{A}$ will have A_ϕ structured as $\mathbf{C}$, A_v structured according to the four-valued Boolean lattice below, and $obs_{\mathbf{A}}$ will map each element to its equivalence class.



Similar constructions apply to the remaining cases, with corresponding two-valued (for θ_2 and θ_3) and single-valued (for θ_4) Boolean lattices as visible domains.

4. Revisiting da Costa algebras

The main gist of our work is to draw a connection between da Costa algebras and $\mathcal{C}_1$ at the light of the behavioral theory of AAL. Since we already have all the necessary

ingredients, in this section, we will focus our attention on understanding the precise connection between the class of da Costa algebras and the class $\mathbb{K}_{\mathcal{C}_1}$ of two-sorted algebras, the $\Gamma_{\mathcal{C}_1}$-behavioral algebraic counterpart of the logic. The idea is to show that, up to isomorphism, there is a one-to-one correspondence between the two.

First of all, given an algebra $\mathbf{A} \in \mathbb{K}_{\mathcal{C}_1}$, let us see how we can obtain a da Costa algebra $\mathbf{U_A}$. Let $\mathbf{A} = \langle A_\phi, A_v, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \sim_\mathbf{A}, \neg_\mathbf{A}, obs_\mathbf{A}\rangle$ and consider the algebraic structure

$$\mathbf{U_A} = \langle A_\phi, \mathbf{f_A}, \mathbf{t_A}, \leq_\mathbf{A}, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \neg_\mathbf{A}\rangle,$$

obtained from $\mathbf{A}$ by letting

– $a \leq_\mathbf{A} b$ iff $(a \Rightarrow_\mathbf{A} b) \equiv^{\mathcal{C}_1}_\mathbf{A} \mathbf{t_A}$.

We can prove that this construction indeed returns a da Costa algebra.

Proposition 4. *Given $\mathbf{A} \in \mathbb{K}_{\mathcal{C}_1}$, $\mathbf{U_A}$ is a da Costa algebra.*

Proof. Recall that we can consider a binary relation $\simeq_\mathbf{A}$ on A_ϕ defined by $a \simeq_\mathbf{A} b$ if $a \leq_\mathbf{A} b$ and $b \leq_\mathbf{A} a$. Let us first of all prove that $\simeq_\mathbf{A}$ coincides with behavioral equivalence $\equiv^{\mathcal{C}_1}_\mathbf{A}$. The fact that $\simeq_\mathbf{A}$ is included in $\equiv^{\mathcal{C}_1}_\mathbf{A}$ follows from the fact that $\mathbf{A}$ behaviorally satisfies the conditional-equation (16) of the definition of $\mathbb{K}_{\mathcal{C}_1}$. To prove the reverse inclusion suppose that $a \equiv^{\mathcal{C}_1}_\mathbf{A} b$. Since $\equiv^{\mathcal{C}_1}_\mathbf{A}$ is a $\Gamma_{\mathcal{C}_1}$-congruence relation and $\Rightarrow \in \Gamma_{\mathcal{C}_1}$ we have that $(a \Rightarrow_\mathbf{A} b) \equiv^{\mathcal{C}_1}_\mathbf{A} (a \Rightarrow_\mathbf{A} a) = \mathbf{t_A}$. Therefore we can conclude that $a \leq_\mathbf{A} b$. In the same way we can conclude that $b \leq_\mathbf{A} a$ and so $a \simeq_\mathbf{A} b$.

To see that $\mathbf{U_A}$ is a da Costa algebra we just have to prove that it satisfies all the conditions on the definition of a da Costa algebra. This is precisely the point where behavioral reasoning comes into play. Verifying that $\mathbf{U_A}$ satisfies a condition of the form $a \leq_\mathbf{A} b$ is, by definition, the same as proving that in $\mathbf{A}$ we have that $(a \Rightarrow_\mathbf{A} b) \equiv^{\mathcal{C}_1}_\mathbf{A} \mathbf{t_A}$. Now, using Proposition 1, it can be easily proved that $\mathbf{U_A}$ satisfies the conditions in the definition of a da Costa algebra, since they all amount to well-known properties of the consequence relation $\vdash_{\mathcal{C}_1}$. □

Now, given a da Costa algebra $\mathbf{U}$, let us see how we can obtain from it an algebra $\mathbf{A_U} \in \mathbb{K}_{\mathcal{C}_1}$. Let $\mathbf{U} = \langle U, 0, 1, \leq_\mathbf{U}, \wedge_\mathbf{U}, \vee_\mathbf{U}, \Rightarrow_\mathbf{U}, \neg_\mathbf{U}\rangle$ be a da Costa algebra. Consider now the two-sorted algebra

$$\mathbf{A_U} = \langle A_\phi, A_v, \wedge_\mathbf{U}, \vee_\mathbf{U}, \Rightarrow_\mathbf{U}, \sim_\mathbf{U}, \neg_\mathbf{U}, obs_\mathbf{U}\rangle$$

obtained from $\mathbf{U}$ by letting

– $A_\phi = U$;
– $A_v = U_{/\simeq_\mathbf{U}} = \{[a]_{\simeq_\mathbf{U}} : a \in U\}$, where $b \simeq_\mathbf{U} c$ if $b \leq_\mathbf{U} c$ and $c \leq_\mathbf{U} b$;
– $obs_\mathbf{U}(a) = [a]_{\simeq_\mathbf{U}}$;
– $\sim_\mathbf{U}$ be derived using the abbreviation of classical negation.

Before we prove that $\mathbf{A_U}$ is an algebra in $\mathbb{K}_{\mathcal{C}_1}$, a few remarks are due. To start with, note that in a da Costa algebra $\mathbf{U}$ the equivalence relation $\simeq_{\mathbf{U}}$ is in general not a congruence, namely with respect to the paraconsistent negation $\neg$. Hence, the usual quotient construction cannot be done directly because the non-congruent operation would not be well-defined on the quotient set $U_{/\simeq_{\mathbf{U}}}$. Using our two-sorted behavioral approach we can, nevertheless, simulate the quotient construction. This is the idea of the construction used to obtain $\mathbf{A_U}$ from $\mathbf{U}$. Moreover, it is an easy exercise to prove that $\simeq_{\mathbf{U}}$ is an equivalence relation compatible with the classical connectives. This observation implies that we can prove the following result.

Lemma 5. *Given an equation $\varphi \approx \psi$ and an assignment h, then*

$$\mathbf{A_U}, h \Vdash\!\vdash^{\Gamma c_1} \varphi \approx \psi \quad \textit{iff} \quad \mathbf{A_U}, h \Vdash obs(\varphi) \approx obs(\psi).$$

Proof. The fact that $\mathbf{A_U}, h \Vdash\!\vdash^{\Gamma c_1} \varphi \approx \psi$ implies $\mathbf{A_U}, h \Vdash obs(\varphi) \approx obs(\psi)$ is obvious since $obs(x)$ is an experiment. On the other direction assume that $\mathbf{A_U}, h \Vdash obs(\varphi) \approx obs(\psi)$. Note that, since $\simeq_{\mathbf{U}}$ is compatible with the classical connectives, we have that $\gamma(h(\varphi), c_1, \ldots, c_n) \simeq_{\mathbf{U}} \gamma(h(\psi), c_1, \ldots, c_n)$ for every γ build up with classical connectives and every $c_1, \ldots, c_n \in A_\phi$ and thus, by definition of $obs_{\mathbf{U}}$, we have that $obs_{\mathbf{U}}(\gamma(h(\varphi), c_1, \ldots, c_n)) = obs_{\mathbf{U}}(\gamma(h(\psi), c_1, \ldots, c_n))$. This fact immediately implies, by definition of behavioral equivalence, $\mathbf{A_U}, h \Vdash\!\vdash^{\Gamma c_1} \varphi \approx \psi$. □

We will now show that $\mathbf{A_U} \in \mathbb{K}_{\mathcal{C}_1}$.

Proposition 6. *Given a da Costa algebra $\mathbf{U}$, $\mathbf{A_U} \in \mathbb{K}_{\mathcal{C}_1}$.*

Proof. First of all note that $obs_{\mathbf{U}}$ is surjective by construction. So, we are left to prove that $\mathbf{A_U}$ behaviorally satisfies all the equations and conditional-equations (1–16), and satisfies the conditional-equations (17-20) in the definition of $\mathbb{K}_{\mathcal{C}_1}$. Using Lemma 5 and the definition of $\mathbf{A_U}$ it is not hard to conclude that, given an equation $\varphi \approx \psi$ and an assignment h, we have that $\mathbf{A_U}, h \Vdash\!\vdash^{\Gamma c_1} \varphi \approx \psi$ iff $\varphi^h_{\mathbf{U}} \simeq_{\mathbf{U}} \psi^h_{\mathbf{U}}$. Moreover, in every da Costa algebra $\mathbf{U}$, we have that $(a \Rightarrow_{\mathbf{U}} b) \simeq_{\mathbf{U}} 1$ is equivalent to having $a \leq_{\mathbf{U}} b$. Using these observations, verifying that $\mathbf{A_U}$ behaviorally satisfies (1–16) amounts to checking well-known properties of every da Costa algebra. Almost all of these properties are proved, for example, in Proposition 1 of (Carnielli *et al.*, 1984). The fact that $\mathbf{A_U}$ satisfies (17–20) is an immediate consequence of the already mentioned fact that $\simeq_{\mathbf{U}}$ is compatible with the classical connectives. □

We now prove that these two constructions are inverse of each other, up to isomorphism.

Theorem 7. *Let $\mathbf{U}$ be a da Costa algebra, and $\mathbf{A} \in \mathbb{K}_{\mathcal{C}_1}$. Then, we have that*

- $\mathbf{U}_{\mathbf{A_U}} = \mathbf{U}$*; and*
- $\mathbf{A}_{\mathbf{U_A}}$ *is isomorphic to* $\mathbf{A}$.

Proof. Let us see that $\mathbf{U}_{\mathbf{A}_\mathbf{U}}$ and $\mathbf{U}$ are in fact equal. Recall that given $\mathbf{U} = \langle U, 0, 1, \leq, \wedge_\mathbf{U}, \vee_\mathbf{U}, \Rightarrow_\mathbf{U}, \neg_\mathbf{U}\rangle$ and applying the construction we obtain $\mathbf{U}_{\mathbf{A}_\mathbf{U}} = \langle U, 0, 1, \leq_{\mathbf{A}_\mathbf{U}}, \wedge_\mathbf{U}, \vee_\mathbf{U}, \Rightarrow_\mathbf{U}, \neg_\mathbf{U}\rangle$. So, all that remains to prove is that $\leq_\mathbf{U}$ and $\leq_{\mathbf{A}_\mathbf{U}}$ coincide. For the purpose, observe that given $a, b \in U$ we have the following sequence of equivalent statements: $a \leq_{\mathbf{A}_\mathbf{U}} b$ iff $(a \Rightarrow_{\mathbf{A}_\mathbf{U}} b) \equiv^{\mathcal{C}_1}_{\mathbf{A}_\mathbf{U}} \mathbf{t}_{\mathbf{A}_\mathbf{U}}$ iff $obs_\mathbf{U}(a \Rightarrow_\mathbf{U} b) = obs_\mathbf{U}(\mathbf{t}_\mathbf{U})$ iff $[a \Rightarrow_\mathbf{U} b]_{\simeq_\mathbf{U}} = [1]_{\simeq_\mathbf{U}}$ iff $(a \Rightarrow_\mathbf{U} b) \simeq_\mathbf{U} 1$ iff $a \leq_\mathbf{U} b$.

Let us now see that $\mathbf{A}_{\mathbf{U}_\mathbf{A}}$ and $\mathbf{A}$ are isomorphic. Recall that given $\mathbf{A} = \langle A_\phi, A_v, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \sim_\mathbf{A}, \neg_\mathbf{A}, obs_\mathbf{A}\rangle$ and applying the constructions we obtain $\mathbf{A}_{\mathbf{U}_\mathbf{A}} = \langle A_\phi, (A_\phi)_{/\simeq_{\mathbf{U}_\mathbf{A}}}, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \sim_{\mathbf{U}_\mathbf{A}}, \neg_\mathbf{A}, obs_{\mathbf{A}_{\mathbf{U}_\mathbf{A}}}\rangle$. Clearly, by definition, $\sim_{\mathbf{U}_\mathbf{A}} = \sim_\mathbf{A}$. Recall also that $\simeq_{\mathbf{U}_\mathbf{A}}$ coincides with behavioral equivalence $\equiv^{\mathcal{C}_1}_\mathbf{A}$. So, all we have to prove is that $(A_\phi)_{/\equiv^{\mathcal{C}_1}_\mathbf{A}}$ is isomorphic to A_v, and that both $obs_\mathbf{A}$ and $obs_{\mathbf{A}_{\mathbf{U}_\mathbf{A}}}$ respect this isomorphism, in the sense that, if $\pi : (A_\phi)_{/\equiv^{\mathcal{C}_1}_\mathbf{A}} \to A_v$ is the isomorphism and $a \in A_\phi$ then we have that $\pi(obs_{\mathbf{A}_{\mathbf{U}_\mathbf{A}}}(a)) = obs_\mathbf{A}(a)$.

Consider the function $\pi : (A_\phi)_{/\equiv^{\mathcal{C}_1}_\mathbf{A}} \to A_v$ such that $[a]_{\equiv^{\mathcal{C}_1}_\mathbf{A}} \overset{\pi}{\mapsto} obs_\mathbf{A}(a)$. Note that π is well-defined, in that it does not depend on any particular choice of representatives of the equivalence classes. Indeed, if $a, b \in A_\phi$ are such that $a \equiv^{\mathcal{C}_1}_\mathbf{A} b$ then, by Lemma 5, we have that $obs_\mathbf{A}(a) = obs_\mathbf{A}(b)$. Let us now prove that π is indeed a bijection. For proving injectivity, let $a, b \in A_\phi$ such that $\pi([a]_{\equiv^{\mathcal{C}_1}_\mathbf{A}}) = \pi([b]_{\equiv^{\mathcal{C}_1}_\mathbf{A}})$. In that case, we have that $obs_\mathbf{A}(a) = obs_\mathbf{A}(b)$ and, again by Lemma 5, we can conclude that $a \equiv^{\mathcal{C}_1}_\mathbf{A} b$. The surjectivity of π follows immediately from the fact that $obs_\mathbf{A}$ is surjective.

To conclude, we just have to check that $\pi(obs_{\mathbf{A}_{\mathbf{U}_\mathbf{A}}}(a)) = obs_\mathbf{A}(a)$. This is immediate, since $obs_{\mathbf{A}_{\mathbf{U}_\mathbf{A}}}(a) = [a]_{\simeq_{\mathbf{U}_\mathbf{A}}} = [a]_{\equiv^{\mathcal{C}_1}_\mathbf{A}}$. □

4.1. *More on valuation semantics*

Having come so far, we must also point at how the bivaluation semantics of (Da Costa *et al.*, 1977) appears naturally in the behavioral context, as a by-product of the behaviorally equivalent algebraic semantics of $\mathcal{C}_1$, similarly to what was achieved in (Caleiro *et al.*, 2003). Still, the result has a deeper meaning here as it can be argued, along the lines of (da Costa *et al.*, 1994), that algebraic valuations are the best behavioral counterpart of logical matrices. A deeper discussion of this topic can be found in (Caleiro *et al.*, 2008b; Gonçalves, 2008). In any case, let us see how to obtain the bivaluation semantics for $\mathcal{C}_1$ directly from the class of algebras $\mathbb{K}_{\mathcal{C}_1}$. This fact also reinforces the idea that the behavioral algebraic approach really captures in an accurate way all the semantical aspects of $\mathcal{C}_1$, and can therefore be seen as an unifying theory. The idea is very simple, the $\mathcal{C}_1$ bivaluations correspond exactly to the observation maps in $\mathbb{K}_{\mathcal{C}_1}$ algebras, given that their visible domain is always a Boolean algebra.

Let $\mathbf{A} = \langle A_\phi, A_v, \wedge_{\mathbf{A}}, \vee_{\mathbf{A}}, \Rightarrow_{\mathbf{A}}, \sim_{\mathbf{A}}, \neg_{\mathbf{A}}, obs_{\mathbf{A}} \rangle$ be a $\mathbb{K}_{\mathcal{C}_1}$ algebra, and recall from Proposition 2 that $\mathbf{A}_v = \langle A_v, \top_{\mathbf{A}}, \bot_{\mathbf{A}}, \sqcap_{\mathbf{A}}, \sqcup_{\mathbf{A}}, \sqsupset_{\mathbf{A}}, -_{\mathbf{A}} \rangle$ is a Boolean algebra. We will now make use of the well-known result that every Boolean algebra is isomorphic to a subdirect power of the two element Boolean algebra $\mathbf{2}$ (Birkhoff, 1967; Burris *et al.*, 1981). For our purposes, it suffices to know that this implies the existence of a set I and an injective homomorphism $\alpha : \mathbf{A}_v \to \mathbf{2}^I$, called a subdirect embedding, such that for each $i \in I$, we have that $\alpha_i : \mathbf{A}_v \to \mathbf{2}$ is surjective.

Clearly, we can view both $L_{\mathcal{C}}$ and A_ϕ as the domains of algebras, $\mathbf{L}_{\mathcal{C}}$ and $\mathbf{A}_\phi$ respectively, over the connectives of $\mathcal{C}_1$. $\mathbf{L}_{\mathcal{C}}$ is simply the free algebra generated from the propositional variables, whereas $\mathbf{A}_\phi$ is the restriction of $\mathbf{A}$ to the sort ϕ. We now consider the following set of functions

$$V_{\mathbf{A}} = \{\nu_{\mathbf{A},i,h} = \alpha_i \circ obs_{\mathbf{A}} \circ h \mid \ i \in I \text{ and } h : \mathbf{L}_{\mathcal{C}} \to \mathbf{A}_\phi \text{ is an homomorphism}\}.$$

Note that for every $i \in I$ and every homomorphism $h : \mathbf{L}_{\mathcal{C}} \to \mathbf{A}_\phi$, we have that $\nu_{\mathbf{A},i,h}$ is a function from $L_{\mathcal{C}}$ to the domain of $\mathbf{2}$, say $\{0, 1\}$. We can collect all the functions of this form in the set $V = \bigcup_{\mathbf{A} \in \mathbb{K}_{\mathcal{C}_1}} V_{\mathbf{A}}$.

Theorem 8. *V is precisely the set of all $\mathcal{C}_1$ bivaluations.*

Proof. First, it can be easily checked that every element of V is a bivaluation for $\mathcal{C}_1$. Then, we have to prove that every bivaluation for $\mathcal{C}_1$ is indeed in V, that is, it can be obtained from some algebra in $K_{\mathcal{C}_1}$. To see this, let ν be a bivaluation for $\mathcal{C}_1$, and consider the two-sorted algebra

$$\mathbf{A}_\nu = \langle A_{\nu,\phi}, A_{\nu,v}, \wedge, \vee, \Rightarrow, \sim, \neg, obs_{\mathbf{A}_\nu} \rangle$$

such that $A_{\nu,\phi} = L_{\mathcal{C}}$, $A_{\nu,v} = \{0, 1\}$ and $obs_{\mathbf{A}_\nu} = \nu$. It is laborious but straightforward to verify that $\mathbf{A}_\nu$ is in fact an algebra in $\mathbb{K}_{\mathcal{C}_1}$. In this case, $\mathbf{A}_{\nu,v}$ is already a (trivial) subdirect product of $\mathbf{2}$. Hence, it must be the case that $I = \{i\}$ is a singleton, and the homomorphism α is the identity. Taking the homomorphism $h : \mathbf{L}_{\mathcal{C}} \to \mathbf{A}_{\nu,\phi}$ also as the identity, it becomes clear that $\nu_{\mathbf{A}_\nu,i,h} = \nu$. □

5. Concluding remarks

In (Caleiro *et al.*, 2009) the authors developed a behavioral generalization of the tools and techniques of abstract algebraic logic in order to extend its scope of applicability, namely to study logics that include non-truth-functional connectives and which are not algebraizable according to the standard approach. In this paper we have used this behavioral approach to study da Costa's $\mathcal{C}$-systems from an algebraic perspective. In particular, we were able to explain the precise connection between da Costa algebras and $\mathcal{C}_1$. As a by-product we were also able to rediscover the usual bivaluation semantics for $\mathcal{C}_1$. Our results are encouraging in that they put an end to the idea that logics like da Costa's $\mathcal{C}$-systems which are not algebraizable in the traditional

sense cannot have a meaningful algebraic counterpart, in a very precise sense. Using our approach, we have given a definitive algebraic justification to the usual view that the $\mathcal{C}$-systems can be understood as being built over a classical Boolean base, then enriched with an extra operator, the paraconsistent negation.

Some work has been devoted in the literature to prove a Stone-like representation result for da Costa algebras (Carnielli *et al.*, 1984; Seoane *et al.*, 1991). We want to make clear, here, that such results are better understood in the behavioral setting we have proposed, as a direct application of the usual Stone representation of Boolean algebras and of our translations between da Costa and $\mathbb{K}_{\mathcal{C}_1}$ algebras, as hidden in the arguments used in the proofs of Theorem 4 in (Carnielli *et al.*, 1984) and of Theorem 7 in (Seoane *et al.*, 1991).

The paper raises several interesting questions. One such question is the connection between our approach and other semantics for the $\mathcal{C}$-systems proposed in the literature, namely possible translation semantics (Carnielli, 2000) and non-deterministic matrices (Avron, 2005). In particular, it would be interesting to study the connection with a generalization of the notion of algebraizable logic in the realm of possible translation semantics introduced in (Bueno-Soler *et al.*, 2005). Another path of future work is the development of the work started in (Caleiro *et al.*, 2008b) on an algebraic perspective of valuation semantics. This seems a promising generalization of matrix semantics that is able to deal with non-truth-functionality in an algebraic way. Since our construction can clearly be generalized to other non-truth-functional logics, it would be very interesting to study other examples to which an algebraic counterpart in the spirit of da Costa algebras could be obtained.

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